INTRODUCING ALGEBRA THROUGH THE GRAPHICAL REPRESENTATION OF FUNCTIONS: A STUDY AMONG LD STUDENTS

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Abstract

This longitudinal study evaluates the impact of a new Algebra 1 course at a High School for language-based learning-disabled (LD) students. The new course prioritized the teaching of relationship graphs and functions as an introduction to algebra. Across three studies, the dissertation documents and evaluates the progress made by LD high school students in Algebra 1 on the interpretation and production of relationship graphs.

The first two studies examine intervention and control students' answers to a written assessment on time-distance graphs, immediately after and two years after the intervention. The third study, examines other intervention and control students' written assessment answers on graphs relating variables other than time and distance, two years after the Algebra 1 course.

Data from the first study show significantly better results for the intervention group in the post-assessment results and significantly better results for the intervention group's post-assessment results in comparison to the control group. Results for the control students taking Algebra 2 were similar to those of the intervention group's pre-assessment results. The second study, again, shows significantly better results for the intervention group two years later. In the third study, even though the intervention group performed slightly better than the control group, the difference between groups was not significant, with both groups showing high levels of performance. Because discussions in the entire department aimed at including a multi-representational view of algebra at all levels, this may have led to changes among the Algebra 2 teachers' views and ways of teaching algebra.

This set of results supports the view that beginning Algebra 1 with instruction on the graphical representation of functions, instead of focusing on the alpha-numeric representation of algebra, allows for learning among LD students that remains over the years.

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Table of Contents

List of Figures

List of Tables

Chapter 1: Introduction

This dissertation aims at evaluating the impact of a new Algebra 1 course at a school for language-based learning disabled (LD) students. The new course represented a curriculum change in the school – it prioritized instruction on the graphical representation of algebra starting with teaching of relationship graphs as an introduction to algebra. Across three studies, the dissertation documents and evaluates the progress made by LD high school students in Algebra 1, focusing on how and what they learn about interpreting function graphs, considering rate and directionality. In two studies we examine how, in comparison to a control group, a first group of participating students learn and retain what they have learned during an Algebra 1 intervention, immediately after, and two years after the intervention, in an assessment interpretation and production of time distance graphs. In a third study, we examine the achievement of a second group of LD students, who also were taught according to the new curriculum, in the production and interpretation of graphs relating variables other than time and distance, two years after the Algebra 1 course.

The research questions guiding the study are:

a. Can high school Algebra 1 LD students learn to interpret function graphs considering rate and directionality?
b. How do they compare to control LD students at a higher grade and age level immediately after the intervention?
c. Did they retain their ability to interpret function graphs two years after they participated in the intervention?
d. How do they compare to control students at the same grade and age level two years after the intervention?

e. How do they interpret and produce graphs relating different kinds of variables?

Rationale for the Study

In high school, students are expected to complete at least a study of Algebra 2 in order to be deemed college ready. While for some this is an easily obtained goal, for many others, the study of algebra is an arduous process; one filled with abstractness, confusion and disconnect from their daily life. A high school Algebra 1 course has all of the potential to change this perspective. This time in a young person's life is a time for them to connect all of the lessons of previous years and build a strong foundation on which to place their high school study of mathematics. This dissertation will look at a shift in the study of Algebra 1 and will mainly look through the eyes of the Algebra 2 students, as this is where the change should ultimately be observed.

The need for proposing and evaluating better approaches to teaching algebra is justified, as described next, because algebra involves multiple representations, algebra as alpha-numeric representation is confusing, and the traditional Algebra 1 curriculum puts the alpha-numeric representation first. We will discuss each of these points next.

Algebra is multi-representational. A true understanding of the complexity of algebra requires that students assimilate many ideas and representations (graphical, tabular, verbal, alpha-numeric). Each representation has its own value and inherent limitations - it is the total picture that gives students a flexible, fluent, comprehensive understanding of algebra.

Current views on the teaching and learning of algebra call for a functional approach to algebra (Schwartz & Yerushalmy, 2003), with an emphasis on the graphical representation of functions, as opposed to the traditional focus on solving equations. Much research on the teaching and learning of algebra from elementary to high schools shows that curriculums rich with links between representations help students gain better access to the content of algebra (Brizuela & Earnest, 2008; Carraher, Schliemann, & Schwartz, 2008; Chazan, 2000; Knuth, 2000; Laughbaum, 1999, 2009; Moschkovich, 1996, 1998; Yerushalmy, 2001, 2006). Studies on algebra among LD students, however, are rare (Geary, 1993; Fuchs & Fuchs, 2002; Keeler & Swanson, 2001; Miles & Forcht, 1995). We need to develop studies to evaluate the contribution of an introductory algebra curriculum (Algebra 1) based in multi-representational thinking to the learning of algebra among LD students.

"Functions and graphs represent one of the earliest points in mathematics at which a student uses one symbolic system to expand and understand another (e.g., algebraic functions and their graphs, data patterns and their graphs, etc.)" (Leinhardt, Zaslavsky & Stein, 1990, p. 2). Studying functions in Algebra 1 allows different, linked representations to be used to increase students' understanding of algebra. Studying functions early also allows the instructor to bring in the strong visual elements that are inherent in the graphical representation.

> Visual recognition of problem or situation is the primary, and most influential, connection to understanding, meaning, properties, uses, and skills related to the problem or situation. In addition, when visualizations are used before any symbolic development, it greatly increases the likelihood that the memory of the mathematical concept being taught will survive (Laughbaum, 2007, p.1).

Students should experience an introduction to algebra that includes visualization, as this may increase student understanding and retention of the symbolic manipulations of algebra. While some of this early insertion of multi-representational thinking about algebra is being proposed by researchers and teaching standards (NCTM, 2000), it is not yet the norm for the majority of Algebra 1 students. This dissertation looks at the impact of a curriculum that gives LD students initial exposure to the graphical representations of functions through discussions on relationships.

Algebra as alpha-numeric representation is confusing. When we approach algebra from a purely alpha-numeric perspective, we create unnecessary boundaries for students. Research has shown that the alpha-numeric representation is difficult for many students and especially difficult for LD students. The equal sign and the use of variables are two of the most basic aspects in the alpha-numeric representation of algebra. Booth (1988) and Kieran (1981) both describe the difficulty students have with the different uses of the equal sign. Also, Filloy, Rojano and Solares (2010) discuss students' difficulties operating on unknown values and feeling as if answers with variables were not viable answers, making it clear that those students weren't comfortable with the use of the variable.

Research with LD students shows that the level of detail involved in working with the alpha-numeric representation is difficult due to challenges these students face in terms of cognitive load, working memory, and retention of rote facts (Geary, 1993, Fuchs & Fuchs, 2002). So in the case of solving an equation such as $3x + 7 = 4x – 9$, if the first step is to subtract 7 from each side of the equation; the pull on working memory to recall (or more often recreate) the facts may cause the student to forget what they were doing in the first place. They may get the computation answer but limitations on working memory cause them to forget what drove them to that calculation. The abstractness of the symbols also detracts from the reason for doing the exercise and LD students are often left feeling that they are merely moving symbols around on the page.

Traditional Algebra 1 curriculum puts the alpha-numeric representation first. When I began teaching high school mathematics 20 years ago with a population of LD students, the curriculum was a traditional Algebra 1 curriculum (see Figure 1). Because of the varied backgrounds of the students coming from different cities, states and even countries, it was necessary to begin at the beginning. The first few chapters of the curriculum were meant to be a review but we spent many weeks introducing and comprehending the concept of negative numbers and variables, it was unusual at the time for these students to have been introduced to integers prior to the study of Algebra 1. Following the progression of this text, students will solve equations, work with inequalities and simplify polynomial expressions before they even think about representing any of these ideas graphically. It is not until Chapter 10 (if they get to Chapter 10) that they will see graphing.

- Ch. 1: Intro to Algebra
- Ch. 2: Operations on Real Numbers
- Ch. 3: Solving Equations
- Ch. 4: Applying Equations
- Ch. 5: Inequalities and Absolute Value
- Ch. 6: Powers and Polynomials
- Ch. 7: Factoring Polynomials
- Ch. 8: Rational Expressions
- Ch. 9: Applying Rational Expressions
- Ch. 10: Relations, Functions and Variables
- Ch. 11: Analytic Geometry
- Ch. 12: Systems of Linear Equations
- Ch. 13: Radicals
- Ch. 14: Quadratic Equations and Functions

Figure 1: Traditional Algebra 1 text chapter topics. List of chapter headings from Holt Algebra 1, Nichols, 1992

Though varying in their degree of understanding, all of the students that come to the high school today have some experience with variables and integers. This is due to an intentional shift in mathematics curriculum of the lower grades (NCTM, 1988, 2000). Additionally, by the time students get to high school, most students have solved linear equations and have had some experience graphing lines. If a course of instruction follows the traditional Algebra 1 curriculum, this means that the first chapter on introducing algebra vocabulary can be skipped (or summarized) and the class can get to graphing sooner. However, as the table of contents in

Figure 1 suggests, graphing does not appear in the curriculum until Chapter 10, so perhaps, at the start of the 4th quarter the class will see a graph. Yet getting to graphing at this time of the year does not pique interest or generate conversation – students treat it as abstractly as they had treated the alpha-numeric approach to solving equations.

A new approach to the Algebra 1 curriculum. The Standards for Curriculum and Instruction by the National Council of Teachers of Mathematics (NCTM, 2000) called for a more connected math program. However, published curricula and not even the NCTM Standards have specific proposals for re-conceiving Algebra 1. It is in the work of Judah Schwartz and Michal Yerushalmy that new ideas are put forth when they state that we need a "new approach to the subject that is based on the centrality of the function and is deeply rooted in the use of multiple linked representations" (Schwartz & Yerushalmy, 2003, p. 41). The idea is relatively simple: change the position of the sections on graphing within the curriculum to make it more central to the conversation. With this recommendation and the recommendations of many other mathematics educators, as well as the dedication of a core group of teachers, we changed the Algebra 1 curriculum, adopting the sequence discussed in detail in Chapter 3: Intervention Curriculum.

The year that the data from the first study was collected began as usual with a review of operations on real numbers and the real number system. But in the new algebra curriculum, we added a chapter on relationships in conjunction with this first chapter. It felt necessary to fully introduce students to the graphical representation and to ground algebra in conversations relating to varying quantities. Studying relationships became the unit in which the tone for developing vocabulary and explaining reasoning was established. We then studied functions, which was typically a topic for the latter part of the year in the old curriculum. Inserting it early in the curriculum allowed us to explore many functions: linear, quadratic, exponential, square root and linear absolute value, allowing students to develop a wider view of algebra. Then after working with the graphical and tabular representations of these functions, we narrowed the focus to linear functions. This gives the students a perspective on how to place linear functions within a family of functions. We then solved equations and inequalities; because we had already graphed functions, we had a visual way to check our work besides plugging the answer back into the equation to see if the sides have the same value, which links very nicely to conversations around systems of linear equations. After exhausting our work with linear equations, we focused on the manipulations of monomials and polynomials. Because we had done some graphing work, we could see the effect on the graph of adding as compared to multiplying polynomials. This gives another view into why $3x^2 + 5x^2$ is $8x^2$ and not $8x^4$. Then we rounded out the year with a study of quadratics. We didn't necessarily cover more ground, but students had a linked multi-representational view of the topics that were discussed.

Dissertation Goals and Overview

The goal of the three studies in this dissertation is to evaluate a small piece of the impact of the new curriculum on student learning, by analyzing how students interpret function graphs, considering rate and directionality, before and after taking the new Algebra 1 course, in comparison to control groups who had taken Algebra 1 as prescribed by the old curriculum.

The impact of the new Algebra 1 course was evaluated through a series of three studies.

Study 1 documents the students' learning before and immediately after the intervention curriculum. Students in the Algebra 1 modified curriculum were given identical pre- and post-tests on relationships graphs of time and distance. Additionally, students in an Algebra 2 course, who had participated in a traditional Algebra 1 curriculum, were given the same test; their results were compared to those of the Algebra 1 intervention students.

Study 2 looks at the same group of students from the initial study, two years later, as they were taking their Algebra 2 course. Using the same test, the students who were part of the original intervention curriculum are compared with their classroom peers who did not take Algebra 1 at this school. For these control students, the Algebra 1 curriculum they were exposed to is unknown.

Study 3 examines yet another group of Algebra 2 students, who had taken the Algebra 1 modified curriculum, now the standard teaching at the school, looking at their responses to relationship graphs that pertain to comparisons other than distance and time. This group of Algebra 2 students is also compared to a control group of students who did not take Algebra 1 at this school.

Relevance of this Dissertation

Our analyses focus on the introduction of algebra as a study of relationships, looking at how LD students interact with this material. They constitute a longitudinal analysis of the data, seeing not just the short-term effect of a curriculum change, but also the long term effect of a specific aspect of the curriculum change.

This particular piece of algebra is important as it represents a new layer to the Algebra 1 curriculum, one that did not exist previously at this level. When I reflect on previous years of teaching Calculus, we began the year talking about rate of change and interpreting relationship graphs so that we could gain entry into talking about the derivative. So the logical questions going into a curricular change are:

- Is there a reason we should hold this discussion until students are ready to study Calculus?

- Can younger students gain access to conversations about relationships and functions at an earlier grade?
- Will the language limitations of the LD student be inhibitive to this goal?
- Will the learning about function graphs remain or will students need to return to the topic each year?

This dissertation rests on the shoulders of much current research on how students interact with specific aspects of the algebra curriculum. It is unique in that it involves a very specific subset of the general population and it resulted in a complete change in the study of Algebra 1 at this particular school. The studies in this dissertation document the progression in understanding of one small part of the curriculum (relationships), which did not previously exist within this school's study of Algebra 1, and does not typically exist in most traditional approaches to Algebra 1.

Chapter 2: Review of the Literature

Algebra represents a new literacy for today's job market and is seen as a gatekeeper to future success (Kaput, 1999; Moses, 2001; Gamoran & Hannigan, 2000; Katz, 2007; Lacampagne, Blair, & Kaput, 1995). "[C]itizenship now requires not only literacy in reading and writing but literacy in math and science" (Moses, 2001 p. 12). Algebra is the cornerstone to more advanced mathematics as well as a key instrument in our informational age.

How best to teach algebra to students is a quest of many mathematics educators. The most common approach is to expose students primarily to the symbols and manipulations of algebra. There are other, more dynamic representations of algebra, but these are not placed first in traditional curriculum (McDougal & McDougal, 2011; Glencoe, 2009). They are saved for later chapters and, in the experience of the author, by then students have lost interest in mathematics. When the study of alpha-numeric manipulation is put first we do not captivate student interest in the subject and lead them to believe that alpha-numeric symbol manipulation is all that algebra is about. When the study of algebra does not include graphical representations from the start, students are not motivated to look at a different view of algebra later, as they already have one way to work with it – through alpha-numeric symbols. With the advent of computer graphing systems, these dynamic representations of algebra become less labor intensive and can be brought into the classroom earlier. Placing the more dynamic and engaging representations of algebra first in a student's course timeline engages and entices algebraic thinking, and saves the alpha-numeric manipulations for a later time, when students realize that they are needed to find more accurate or precise answers. This approach would give students a reason for wanting to know those manipulations and a context in which to place them.

This chapter looks broadly at why a study of algebra is important and at the documented difficulties that LD students face in their exploration of the subject. We will describe and discuss students' difficulties with algebra as the basis for the argument against beginning algebra with the alpha-numeric representation. This will be followed by a description of the functional approach to algebra that has more recently evolved in the literature and by an argument for promoting this approach among LD student populations.

Why is Algebra so Important?

Moses (2001), founder of The Algebra Project, argues that technology brings abstract symbolic representations front and center and algebraic representations are the tools that control technology:

> [I]n order to use this technology to organize work you have to understand these symbolic representations and the place that society has assigned for young people to learn this symbolism – this is algebra. So now algebra becomes an enormous barrier. (Moses, 2001 p. 13)

Without adequate instruction in algebra, students are denied access to futures to which they may aspire in an increasingly technological society. Moses also points out that, in the United States, algebra is the entry point to advanced mathematics and quantitative literacy (Richardson, 2007).

"Algebra for Everyone" is the slogan of reform from the early 1990s (Edwards, 1990). Many of the authors of the papers collected in the Algebra Initiative Colloquium (Lacampagne, Blair, Kaput, 1995) support the idea that, in order to be educated in today's society, one must have a minimum of a ninth grade understanding of algebra. Twenty years later, states across the country are continuing to struggle with mandates that require students to complete a college preparatory algebra course before graduating high school (Gamoran & Hannigan, 2000). Many students are still struggling with and failing at algebra. In addition, many states require passing a standardized test that incorporates knowledge of algebra to receive a high school diploma. As expressed by Schoenfeld:

> There is a new literacy requirement for citizenship. Algebra today plays the role that reading and writing did in the industrial age. If one does not have algebra, one cannot understand much of science, statistics, business, or today's technology. Thus, algebra has become an academic passport for passage into virtually every avenue of the job market and every street of schooling. With too few exceptions, students who do not study algebra are therefore relegated to menial jobs and are unable often even to undertake training programs for jobs in which they might be interested. They are sorted out of the opportunities to become productive citizens in our society. (Schoenfeld, 1995, p. 11)

The study of algebra was traditionally held until high school or late middle school while the earlier grades focused on arithmetic. Kieran (2006) describes major themes that have emerged over the thirty-year (1977-2006) history of research on algebra, within the field of Psychology of Mathematics Education. In these 30 years discussions involved the transition from arithmetic to algebra, helping students understand variables and unknowns, equations and equation solving, as well as using algebra to solve word problems. In the mid 1980s discussions around technological tools emerged with a focus on multiple representations. From the mid 1990s conversations around algebraic thinking in elementary schools began to emerge. Research is showing that understanding arithmetic and algebraic thinking do not have to be distinct achievements (Schliemann, Carraher, & Brizuela, 2007). As the landscape of when and how algebraic ideas should be introduced has changed, so has the research and proposed approaches to algebra.

With the call for increased STEM (Science, Technology, Engineering, and Mathematics) education in our schools, perhaps this is less of an argument and more of a reminder of why the study of algebra is important. Both of the National Council of Teachers of Mathematics

Standards documents (1989, 2000) asked for students to analyze patterns of change in different representations, a classic algebraic-type skill. The National Science Education Standards (National Research Council, 1996) asks for students to use mathematical functions to identify patterns in data (Carlson, Jacobs, Coe, Larsen & Hsu, 2002). Additionally, the current Common Core Standards for Mathematics (2010) includes "functions" as its own category on the grade standards starting in 8^{th} grade, making algebra at the secondary level necessarily related to functions.

Algebraic ideas are also making their way into the elementary and middle school curricula. At the start of my teaching career, students arrived in my high school algebra class with no knowledge of integers or variables. Today, nearly all students entering high school have solved an equation with one variable, have an understanding of the concept of variable, and have knowledge of integers. While the students' proficiency with these topics varies greatly, nearly all of the students entering secondary school have solved an equation and worked with the concept of a variable. In this transition time, much filling in, connecting concepts and correcting misunderstandings, as well as introducing new ideas and conceptual understandings of how algebra should look or fit into its mathematical environment must be done. Learning from the challenges students face and moving those conceptual ideas to the middle school and even elementary school curriculum allow for students to remain open to the ideas yet to come.

However, simply stating that it is desirable for all students to study algebra is not sufficient. Creating a different approach to algebra that is accessible to all students, even to those for whom understanding mathematical concepts is a challenge, is what is necessary. Rethinking the approach to algebra in its scope, sequence, focus and introduction is needed to actualize "algebra for everyone."

What Challenges do LD Students Face in Mathematics?

The teaching and learning of algebra can be challenging among LD students who are restricted by their limitations on cognitive load, working memory, retention of rote facts, and ability to mediate the metacognitive acts of problem solving (Fuchs & Fuchs, 2002; Geary, 1993). Specific difficulties with auditory or visual processing are also factors that contribute to difficulties with mathematics and learning in general (Keeler & Swanson, 2001; Wilson & Swanson, 2001). This is an important area for new studies because the needs of the special education community in the midst of mathematics reform have been widely unaddressed (Woodward, 2004). By analyzing and reflecting on this specific population, much can be learned about how LD students are interacting with the curriculum and where their difficulties lie.

Teaching algebra to the LD population brings in a host of challenges related to the learning disability and the way in which these students process information. Having a learning disability does not predetermine a student's success or challenge in mathematics (Bryant, Bryant & Hammill, 2000), but, LD students who are not specifically identified with a math disability have general difficulties that are directly related to mathematical learning, such as:

- Misreading computation signs
- Having difficulty with the language of math
- Taking a long time to complete calculations
- Counting on fingers
- Jumping impulsively into arithmetic operations
- Skipping rows or columns when calculating
- Experiencing difficulties with the spatial arrangements of numbers
- Having difficulty learning to tell time (Bryant, et al., 2004, p. 173).

LD students' difficulties, as described by Bryant, et.al. (2000), show an impact on a their ability to manipulate alpha-numeric symbols before we even discuss any conceptual challenges. They found that LD students often jump impulsively into arithmetic operations or misread computation signs. Oftentimes, as in the equation $3x + 7 = 4x – 9$, the student may choose to subtract 7 from each side, but does not attend to the negative sign in front of the 9, and ends up with $3x = 4x + 2$. If algebra and arithmetic are primarily approached by manipulating symbols, LD students may find this particularly limiting, as they have difficulty with basic computation and with spatial arrangements of numbers and symbols, both of which are important when working with algebra equations. Without opportunities to express their ideas in different formats or without an emphasis on the conceptual understanding, many LD students are perceived as not understanding a particular topic when their written output is plagued with errors due to the challenges of their LD.

Cawley, Parmar, Yan, and Miller (1996), in a study of arithmetic skill, found that LD students acquire skills in broken sequences rather than a steadily increasing pattern, making it necessary to repeat and revisit ideas often. In addition, they found that LD students have lower retention rates especially as concepts become more complex. Research by Keeler and Swanson (2001) also supports the "link between working memory and math performance and further suggests that one factor that may influence working memory is the knowledge of memory strategies" (p. 433). Approaching algebra solely from a traditional algebraic perspective, focused on equations and alpha-numeric manipulation, will further limit these students as this method is predicated on rigid rules and procedures, which are difficult for LD students to memorize.

Another study by Miles and Forcht (1995) further supports that many LD students demonstrate difficulties when they first encounter algebraic concepts because of the alpha-numeric or abstract reasoning involved. The recommendations by Witzell, Smith, & Brownell (2001) have been to initially use hands-on, manipulative, and pictorial representations to help work with the abstract symbols of algebra. This supports the use of a multi-representational strategy for teaching algebra.

Additionally, LD students benefit from direct strategy instruction (vanGarderen, 2007; Ives & Hoy, 2003). For example, vanGarderen (2007), in her assessment and strategy intervention of LD students using diagrams to solve word problems, found that “prior to receiving instruction, the students rarely if at all used a diagram to solve a word problem … The most disconcerting finding was the lack of knowledge these eighth grade students had about what a diagram is and how to generate a diagram” (p. 550). Although “draw a diagram” is one of the essential steps in a problem solving process, vanGarderen (2007) found that the majority of diagrams these students drew were pictorial in nature and not schematic. Direct strategy instruction in using diagrams was found helpful to these students who, at the end of the instruction, met the criteria for mastery of solving one and two-step word problems, whereas prior to instruction they fell below the baseline in performance on these types of problems. In addition to their better performance on specific tasks, the students began to generate diagrams for all problems. After the instruction, students’ drawings of diagrams continued to be schematic in nature and did not regress in non-routine word problems. Instruction “with the emphasis on conceptual understanding of a diagram, how to generate different types of diagrams and how to use a diagram as a tool to solve word problems” (vanGarderen, 2007, p. 552) helped LD students internalize some problem solving skills. Results of this study support the conclusion that LD students need opportunities to interact with and practice using different representations in order to explore their interconnectedness. It also demonstrates that students internalize concepts better when conceptual understanding is paired with direct strategy instruction.

It is not only essential to provide LD students with clear content instruction, it is necessary to provide these students with clear and innovative strategy instruction as well. My experience has taught me that a different approach to algebra is needed for LD students. By concurrently exploring the alpha-numeric and graphical representation of functions with the graphical representation introduced first, we may allow all students, regardless of their arithmetic fluency, access to algebraic concepts. This is a model that should not lose its usefulness as LD students advance their study of mathematics beyond Algebra 1.

The Barriers to Algebra

Studying algebra through its alpha-numeric representation gives students the perception that algebra is rigid, precise and unforgiving in its notation, creating an initial barrier for many students. Difficulties that relate to the varying use of the notation and its subtle implied contextual cues are what make algebra study through the alpha-numeric representation so difficult. Even when the context of solving a linear equation in its alpha-numeric representation is understood, the steps to solution may be plagued with arithmetic errors and, in a traditional approach to algebra, the only way to check the accuracy of a solution is through arithmetic means. When algebra is presented only through its alpha-numeric representation, students come to view algebra as frustratingly tedious.

Outlined next is some of the research related to the barriers that the notation provides as well as thoughts on how to better interact with the notation.

The equal sign. In its early use in arithmetic, the equal sign appears as a prompt for an answer, such as in $3x + 5x =$. However, in algebra, it is also used as a bidirectional operator with which we compare two expressions: $3x + 7 = 4x - 9$, for example.

When children are in elementary school, the use of the equal sign as a unidirectional operator is prevalent, as in 3 + 4 = 7 (Booth 1988; Kieran, 1981). In this case the equal sign signals an "answer" to a question. Kieran (1981) found that even 12 to 14 year-old students believed that the equal sign was something that preceded an answer. This can be traced back to the elementary school level (such as in 3 + 4 =), where the use of the equal sign to prompt students to write down an answer is prevalent. Booth (1988) found that this conceptualization can persist through high school.

In my work with high school students at the Algebra 1 level, it is evident that many students use this symbol as a unidirectional operator. The first time I typically see this is in the fall term when the class is simplifying numerical expressions such as 5(2) + 4 – 9. Students who do not fully grasp the idea of a mathematical sentence will often "show their work" as 5(2) = 10 + 4 = 14 – 9 = 5. Students who use the equal sign like that are often confused when I say "so what you are saying here is that 5 times 2 is the same as 10 plus 4 which is the same as 14 minus 9 which is the same as 5?" They try to explain that it is what they *did*. "I see that, but if I were to look at this later, what might I think?" This difficulty needs to be corrected before they move on to solving equations, so that they can truly see what is being represented in the equation. Having the opportunity to spend more time working on this level of use of an equal sign before jumping into solving equations is valuable. This may be afforded only when, in "chapter 2" they are not solving equations, and have more opportunities for feedback in their use of the equal sign.

At the beginning of a study of equations, simple problems such as x + 4 = 7 do not necessarily conflict with a student's understanding of the equal sign, as a student may recall the memorized fact from above and tell you the answer is 3. However, with the simple change of removing the safety of the memorized fact, it is my experience that these same students find the problem $x + 3.9 = 6.8$ very challenging. The first problem did not cause the students to find a different solution strategy, they could use their knowledge of rote facts to solve the problem and the question did not cause them to rethink the use of the equal sign. The second problem eliminates their ability to use rote facts and, if a student shows some hesitation with this, it may show that this student has not necessarily understood the use of inverse operations. Moreover, as problems become more complex ($2x + 3 = 4x – 9$ for example), they move further and further from the students' limited understanding of the use of the equal sign. This example can be best understood within the framework of a functional approach to algebra: $2x + 3$ is certainly not the same as $4x – 9$, except for one value of x. At that one value of x, these two expressions will have the same value and $2x + 3 = 4x - 9$. This is certainly a different use of the equal sign from $3 + 4 =$. A third use of the equal sign that is prevalent at the elementary level is in formulas, for example $A = lw$. When not pointed out to some students, the subtleties of the application of this simple symbol may be a source of confusion. Exposing students to different uses of the equal sign and giving students opportunities to interact, earlier in their math experience, with the different meanings of the equal sign may create less confusion in later years. Examples such as $7 = 3 + 4$ (Booth, 1988) or, even better, $5 + 2 = 3 + 4$, would give students the opportunity to see this symbol as something other than a prompt for an answer. In addition, providing opportunities to point out to students the varying uses of the equal sign can be a productive classroom discussion.

Use of letters as generalized numbers or variables. The different uses of variables, such as variable as unknown, as a generalized number, or as a functional variable, play a significant role in students' understanding of algebra. The multiple notions of variable cause different levels of difficulty for students (Kuchemann, 1978; Usiskin, 1999). Similar to the challenges with the equal sign, the variable is treated differently in different circumstances. For instance, in $A = lw$ the variables have the feel of knowns (Usiskin, 1999). In the functional relationship of $y = 3x$ they have the feeling of varying quantities. And in the example of $12 = 4x$ the variable has a feeling of an unknown value. So, in the case of a particular lesson, students are using the variable in one way and a different way in another lesson. When they are in the midst of a given lesson, students can be persuaded to understand the particular use of the variable, and will make assumptions and gain a feel for how the variable should be used. As classes continue and the use of the variable changes, cognitive conflict and consequently confusion and frustration can result. In addition, in the situation of standardized assessment or cumulative assessment, where questions involving different meanings of the variable are placed next to each other, students will often struggle with the rapid unannounced change in their use. Students should be given the opportunity to explore the use of variables in different contexts as they begin to generate and use variables in different ways. Approaching algebra through a study of functions brings the different applications of variables to the surface when students are asked to look at $3x + 2 = 4x – 1$ as the solution to the system of equations $y = 3x + 2$ and $y = 4x-1$. As students become more and more engaged in different uses of the variable, explicit comparison of subtleties and assumptions students are making about the variable are beneficial.

Focus on finding particular answers. The precision and subtlety of algebraic notation competes with students' experiences with arithmetic and they may be unsettled by an answer that is not a number, as Booth (1984) found in interviews conducted with thirteen to sixteen year olds. In a series of interviews in which students were given problems similar to the one in Figure

2, she found that, while the student danced around the answer 11y, he did not feel like the answer was correct.

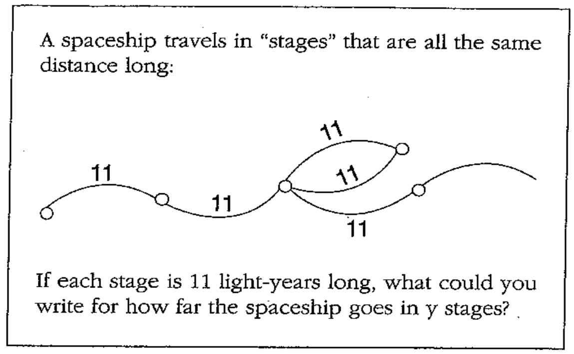


Figure 2: Sample problem Algebra: Children's strategies and errors by Booth, Copyright 1984. Windsor, UK: NFER–Nelson.

The student thought that the response to the answer should be a concrete value and thought that the answer 11 times y was ridiculously simple. Such was this student's response after stating 11 times y: "What, is that all it was? Why didn't you say so? I thought you wanted the answer" (Booth, 1984). A student's inability to see responses with variables as viable answers is another barrier to algebra instruction. Students need opportunities to explore relationships between variables. If Jane is 3 years older than Marge, what could their ages be? What if Marge is x? Setting up relationships between two variables (Jane and Marge's ages) is a productive extension that allows students not only to be receptive to variables, but to use and create their own variables as solutions.

Operating on unknowns. Filloy, Rojano and Solares (2010) found that students have a great deal of difficulty operating on unknowns. In their interviews of fifteen students, ages thirteen to fourteen, in their first year of secondary school, they found that while these students had been successful at solving equations and working with single unknowns, they were not necessarily able to operate on unknowns in such a way as to solve systems of linear equations. Given a novel situation of solving a system of equations, the researchers found that students resorted to non-algebraic meanings and resources, such as solution strategies that focused on arithmetic and mental calculations, concrete representations, or logical inferences.

Sfard and Linchevsky (1994) describe interactions with novice algebra learners who can solve linear equations of the form $ax + b = c$ but are confused when asked to solve linear equations of the form $ax + b = cx + d$. As they describe, students are using arithmetic processes to solve for the missing quantity in the first case, because students can invoke an arithmetic process to solve the equation. Students can think about what number when multiplied by a and added to b results in c. However, in the second case these same students could not conceptualize how to solve for x. As described, one student who "for the first time is faced with a non-arithmetic equation, 5x + 12 = 8x + 47," saw the variable x as two different numbers on different sides of the equal sign (Sfard & Linchevsky, 1994, p. 210). The student said, "We must find two things. Something that when I multiply it or divide it... will be equal to the other thing. One must find the way of making them equal, there are two equations here" (p. 210). Explicit statement of the fact that in an equation the same variable has to represent the same value could perhaps solve this problem of double interpretation (Herscovics & Linchevsky, 1991). What is of note is that this did not inherently occur to this student.

There are many programs that use manipulatives to help students understand the idea of variables on both sides of an equation. I have experience applying these strategies with students and while this does allow them passage into solving linear equations, it does not extend into solving other types of equations, such as quadratics. On the other hand, in my experience, starting with a graph of two linear functions and looking for the point at which they share the same x and y value has been transferable to the larger study of algebra.

Filloy and Rojano (1989) interviewed twelve to fourteen year old students who had a strong ability to solve equations with a variable on one side (e.g. $ax + bx = c$ or $(ax + b)c = d$) but no ready strategy to work with equations with a variable on both sides of the equal sign. They worked with the students with two different concrete models, one involving area, the geometric model (such as the approach used with *Algebra Tiles*), and one involving the balance method (such as is used in *Hands On Equation).* They found that there is a didactic cut at this point, where students move from working with a variable on one side and a variable on both sides of the equations. They learned that with each method:

1. There are ways, specific to each model, of translating equations into terms of the model that become obstacles to the further use of the model
2. Some transferences in the uses of the model are more "natural" in one form of the model than the other. (Filloy and Rojano, 1989, p. 24)

These are models that I have seen in use in classrooms and within strategies that students bring to the classroom. The challenging aspect with these types of modeling of the algebraic process as Filloy and Rojano (1989) point out is that they do not get at what the algebra is intended to achieve. While with either model, students can learn the manipulations of the processes needed to solve an algebraic equation, the models do not link meaningfully back to the concept. Students were not able, on their own, to use the model to achieve the next level of understanding, nor did they apply back to the abstract process of algebra what the models had shown them.

Understanding equivalent transformations on equations. When we break down the process for solving a linear equation in the alpha-numeric representation, we are creating equations with the same solution. The "rule" that allows us to "add the same thing to both sides" for example relies heavily on the properties of equality. Teaching solving linear equations solely through this equivalent transformation process can create a narrow, restricted view of the process.

A study by Steinberg, Sleeman and Ktorza (1990) found that algebra students do not recognize equivalent equations. In a study of ninety-eight 8^{th} and 9^{th} grade students all having received instruction in simplifying algebraic expressions and using transformations to solve simple equations, most did not utilize this understanding to gauge that a simple transformation would give an equivalent equation. Even when the comparison was similar to their first step in a solution, such as in $2x + 3 = 5$ and $2x + 3 - 3 = 5 - 3$, students did not readily recognize the equivalence of these two equations. "These results indicate that many students are not sure that an equation that has been derived by a valid transformation has the same solution, or are unable to recognize when an equation has been transformed in a way that does not alter the answer" (Steinberg, et al, 1990, p. 119). Teaching in this way, students begin to just "do" operations to each side, without really reflecting on what they are doing or the gestalt of the process. This becomes a means to an ends; students learn to make these transformations and in the end it becomes a solution. But the connection between each iteration of the process is a more subtle idea for students to understand, especially in the light of the goal-directed behavior of finding a solution.

An understanding of the equivalence of equations should be explicitly discussed and reflected on by students. Herscovics and Linchevski (1996) did an intervention with six students who were explicitly taught solving equations through creating equivalent expressions and cancellation. For example, $8x + 3 = 4x + 19$ would be turned into $4x + 4x + 3 = 4x + 16 + 3$ and similar terms can be canceled from each side leaving $4x = 16$. While they had success with this approach to teaching solving equations, students were limited by their flexibility with numbers. For instance, some students represented $-6x$ as $-3x + 3x$. So, while students could be taught to conceptualize equivalent equations, fluency with arithmetic played a factor in their ability to find success with this model. Simplification of algebraic expressions creates serious difficulties for many students (Kuchemann, 1981). Therefore relying on students' ability to work with equivalent expressions is also a challenge. A study of algebra that is initially less dependent on computation would remove another barrier to understanding algebra.

In Summary. Research shows that students face many challenges as they approach the algebra curriculum. Some could be solved by restructuring our approach to algebra and leaving open possibilities for the future: for example, in elementary school, not stating that larger numbers cannot be subtracted from smaller numbers. Some may be solved by better understanding students' challenges relating to the use of notation and stating the competing understandings outright.

But, algebra is more than the alpha-numeric representation. The traditional curriculum for algebra focuses mainly on memorizing and internalizing the rules for solving algebra equations. Then, as students progress through higher levels of secondary mathematics, they are shown the connections between the alpha-numeric manipulations and the representation of functions and equations on a coordinate plane. These difficulties with the alpha-numeric representation are exacerbated among LD students. When the algebraic manipulation strategies are placed before the study of the graphical representation, LD students have difficulty making the shift in thinking about algebra from one of manipulating variables and equations to one of manipulating functions. The two ideas seem disjointed and disconnected because, as with the LD students that I have worked with who have received instruction in the alpha-numeric representation before the graphic representation, they would believe that graphing is merely an interesting sideline to the real work of algebra - it is just "something you do." They did not inherently recognize that graphs are representations of algebraic relationships.

Of course, with any approach to algebra, the challenge of notation will still exist within the alpha-numeric representation of the content. The goal, then, is to find other ways to interact with the algebra concepts so that this representation becomes meaningful to the student and is not a barrier to their overall understanding of the subject.

Algebra as the Study of Functions

Function has been identified as a core concept in mathematics and the path to understanding algebra (Oehrtmann, Carlson, Thompson, 2008, Yerushalmy & Schwartz, 1993). The growing body of work related to algebra as the study of functions has focused on distinguishing the differences between approaches to algebra (Bednarz & Kieran & Lee, 1996; Smith, 2008; Yerushalmy, 2006), describing specific aspects of students' work with functions (Carlson 1998; Moschkovich, 1996, 1998; Nemirovsky, 1998), experiences with teaching algebra from a functional approach (Chazan, 2000; Laughbaum, 1999, 2002, 2009; Leinhardt et. al., 1990; Yerushalmy & Schwartz, 1993), and evaluating the impact of introducing a functional approach to algebra in elementary school or high school. Indeed, these studies (Blanton & Kaput, 2008; Brenner & Mayer & Moseley & Brar & Duran & Reed & Webb, 1997; Brizuela & Earnest, 2008; Carraher, Schliemann, & Schwartz, 2008; Chazan, 2000; Knuth, 2000;

Yerushalmy, 2006) have found that a curriculum rich with links between alpha-numeric and graphic representations help students gain better access to the content of algebra.

There is much discussion about the role functions play in the study of mathematics in general and in high school algebra specifically (Beckmann & Thompson & Senk, 1999; Ferrini-Mundy & Lauten, 1993; Schwartz, 1992), and there is a growing realization that being able to move or transfer between representations plays an important role in being able to reason with functions (Knuth, 2000; Porzio, 1999; Wilson & Krapfl, 1994). Student work with the different representations of functions is a key to student understanding of algebra and to success in advanced mathematics (Oehrtman, et al. 2008). This perspective is included in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), which recommends that instructional programs enable students to select, apply, and transfer among mathematical representations to solve problems (Cunningham, 2010). Also the current Common Core Standards (2010) emphasizes the study of functions as central to the high school curriculum.

What is a Functional Approach to Algebra?

How to approach teaching algebra to students has been an active conversation for many years in the mathematics teaching community. In the collection of articles related to different approaches for teaching algebra, Bednarz, Kieran, and Lee (1996) summarized the approaches, depicted in Table 1.

Table 1: Summary of Approaches to Algebra

Type of Approach	Brief Description
Generalization approach	Introducing algebra through patterns (i.e. numeric or geometric patterns) (Mason 1996; Mason & Sutherland, 2002; Stacey & MacGregor, 2001).
Problem-solving approach	Introducing algebra through word problems, focusing on solving equations and finding unknowns (Boero, 2001; Filloy, et. al. 2001).
Modeling approach	Introducing algebra through "mathematical narratives" created to describe events and situations; focusing on variables involved in modeling a situation. (Janvier, 1996; Nemirovsky, 1996).
Functional approach	Focus on developing an understanding of functions in many different representations (table of values, graph of relationship, a rule expressed in algebraic symbols) and the idea of variable. (Confrey, 1992; Yerushalmy & Schwartz, 1993; Kaput, 1999; Laughbaum, 2009; Chazan, 2000).

All of these approaches involve more sense-making than what is typically offered to students in a heavy alpha-numeric, manipulation of variables approach, typical of the traditional Algebra 1 curriculum. It is my experience that the first three approaches (generalization, problem-solving, and modeling), while situating algebra in a meaningful narrative or pattern recognition, rely heavily on the alpha-numeric representation of algebra. It is only the functional approach to algebra that directly links the graphical and alpha-numeric representations of algebra from the very beginning.

In their strong support for the functional approach to algebra, Schwartz and Yerushalmy (1993) argued for an algebra curriculum that is "based on the centrality of the function and is deeply rooted in the use of multiple linked representations" (Schwartz & Yerushalmy, 1993, p.

41). They proposed looking at algebra through a wider lens, one which includes using technology, and beginning the discussion of functions concurrently with (if not earlier than) discussions regarding symbolic manipulation, so that students can speak more generally about classes of functions earlier in their algebraic career. In the traditional approach to Algebra 1 (where students learn to solve equations, solve inequalities, and eventually graph functions, but only linear functions), the learner does not see the variety of applications and the importance of the interconnectedness of the representations of different functions. As Schwartz and Yerushalmy (1993) describe, students then over-generalize properties that only hold for linear functions, so that when other functions are studied, their overgeneralizations do not readily apply and that creates confusion, bringing into question a student's comprehensive understanding of algebra.

Chazan (1993, 1999, 2000) wrote about his experience teaching a "functions-based" approach to Algebra 1, which he contrasted with teaching a traditional approach to Algebra 1 in previous years. Though the student population was quite different in each of the teaching situations, he reflected on the difference in the quality of interactions he was having with his students. Teaching a traditional Algebra 1 course in a private prep school, Chazan laments finding that students were less engaged in the scholarly pursuits of algebra than they were in a Biblical criticism course in which he taught the same students. The students were less engaged in debate and discussion as the algorithm for completing a certain type of problem was presented by him and carried out by the students. He found a different type of engagement of students in his second school, a rural school, where he worked with lower level Algebra 1 students.

> [M]aking functions and their standard representations – new mathematical objects for students – central to the course, changed my experience. This approach helped me express the problems I posed to students in a way that allowed them to understand the desired goals. At the same time, it gave them resources which they

> could use to solve the problems even before being taught standard methods. Standard methods could then be introduced to students as ways of solving problems which they already understood. I felt on less shaky ground with respect to a motive for learning. At the same time, I felt better equipped to help students see the mathematics we were studying in the activity of people they knew, across a range of professions, vocations, and avocations (Chazan, 1993, p. 126).

A functions-based approach allowed students to work with functions of which they had no previous knowledge, affording a different type of classroom experience than the teacher-led discourse conducted in the traditional Algebra 1 classroom.

While Chazan (2000) was not completely pleased with the mathematical outcome of his students, he found that a functions-based approach offered more meaningful interactions with his students than a more procedurally focused curriculum had offered him in the past. Focusing on the function as central to algebra afforded him a different view of algebra. A traditional approach focuses on procedures and pieces that themselves have little meaning and do not speak to the concept of algebra. “Having a characterization of the central object of the course has changed that situation dramatically” (Chazan, 2000, p. 110). By describing algebra through functions, he found that, as a teacher, he gained a greater understanding of the reasons why we study algebra and of the possible applications of algebra. Thus he was able to imagine different paths through the curriculum.

Laughbaum (1999, 2002, 2009) also argues for a functions approach from his experiences with students at the university level who need remedial math classes and continue to struggle with algebraic concepts. He states that approaching algebra as the study of functions “capitalizes on the sense of numbers through the work with functions in numeric form – right from day one” (2009, p. 5). With a functional approach to algebra he has been able to use students’ natural disposition toward patterns and relationships to make meaningful connections between the algebra being studied and the world that they live in. Relying primarily on the hand-held graphing calculator, Laughbaum uses a functions approach to reveal mathematical relationships and rules through a series of explorations and conjecture activities.

Leinhardt, Zaslavsky and Stein (1990) report that, "[f]unctions and graphs represent one of the earliest points in mathematics at which a student uses one symbolic system to expand and understand another (e.g., algebraic functions and their graphs, data patterns and their graphs, etc.)" (p. 2). When students are exposed to these different representations, the instruction should focus on exploring and understanding the interconnected nature of different representations of functions. But as the traditional curriculum focuses on a checklist of skills students can perform, some of the more conceptual aspects of this connection between the graphical and the symbolic representations are lost.

Schwartz (2005) considers the semantic difficulties of the concept of function and stresses that a function is both a process and an action that "transforms or maps a number (or group of numbers) into another number (Schwartz, 2005, p. 1)." A function is also an object to which actions may be done. Thinking of a function in only one representation does not allow the dual nature of the function to become explicit. "While the dual nature of function is central to the concept of function, we believe that the confusion can be largely circumvented by choosing the way we represent functions carefully and using these representations to complement one another in constructive ways" (Schwartz, 2005, p. 3). To avoid difficulties similar to those related to the interpretation of the equal sign, the dual nature of function and inconsistencies that come with thinking about function in two ways should be explicitly stated and explored. By demystifying the subject and pointing out the different ways of thinking about functions and exploring these different ways, the students move ahead in their thinking instead of being bogged down in a semantic misunderstanding. Presenting the concept of function in different ways allows students to understand and develop a more comprehensive notion of functions.

There is a story of three blind men exploring an elephant. Depending on which part of the elephant they were touching, they each had a different understanding of what an elephant was. It is the same with students and function. If students are given the opportunity to explore functions and algebra from only one perspective or one representation, they may have a limited view of what a function is and what algebra is about. By exploring algebra from many different perspectives or representations, students are given a more complete understanding of the overall concept. The different representations of functions (see Table 2) should be used strategically to push and help students question their understanding of functions.

By drawing attention to the dual nature of functions (as process and as object), students could gain a greater insight into understanding functions and working with them flexibly. “The function has many representations – two of the most powerful being the [alpha-numeric] and the graphical. These two representations are cognitively complementary. Each representation offers access to insights that is available only with difficulty in the other representation” (Schwartz & Yerushalmy, 2003, p. 283). The functional approach to algebra links together different representations and explores the benefits and shortcomings of each representation, increasing the student’s proficiency with all of the representations (tables, graphs, alpha-numeric).

Table 2: Summary of Types of Representations of Functions (Schwartz, 2005)

Type of Representation	Brief Description
Iconic	Such as the function machine, showing function as a process
Numerical	Such as creating a table of inputs and outputs; when created from a function machine this is still a process-oriented representation, but when reflected on separate from the calculations, the object qualities of function can be observed
Symbolic	"For polynomials and rational functions, symbolic representations are effective statements of computational procedures… the [alpha-numeric] representation puts the process side of the function forward more saliently" (Schwartz, 2005).
Graphical	Makes the object side of the function more visible. From the graph, you can answer more global questions about the function. Questions about the properties of the function can be answered, such as, is the function always increasing? The benefits of technology are that the graphical representation can be created effortlessly, with some caution as to the choice of window size.
Verbal	Using language to mitigate properties that are noticed in statements such as increasing at a steady rate or increasing faster and faster or increasing slower and slower. "This representation relies on the fact that over a small enough region of its domain, there are only three ways a function of one variable can increase" (Schwartz, 2005). Consequently there are only three ways a function can decrease and one way to remain constant, a limited seven descriptions of functions. With this analysis, the object side of function can be more readily observed.

The emergence of this method is linked to computer software becoming more readily available and easy to navigate. Most desirable software demonstrations are those that allow students to manipulate the function in either representation and see the results in all representations. What is currently available on the open and free market for dynamic systems is very different than what was available even five years ago. It seems safe to assume that future software advancements will only make it easier to bring a functional approach to algebra to the classroom. The underlying theme is the same – linking the multiple representations of table, graph and alpha-numeric symbols creates a more meaningful view of algebra.

Studies on Students Learning about Functions

Different representations allow for different insights into understanding a problem. Carraher, Schliemann and Schwartz (2008), in their work with 3rd to 5th graders, observed that invoking different representations helped young students conceptualize the ideas of variable and equation. For instance, the use of a table to list results helped students grow from seeing individual instances to seeing a patterned relationship and judge correct or incorrect suggestions of an individual incidence. They were able to get students to recognize that there are many possible outcomes related to a given statement, and that only one of them would satisfy the equation. Moreover, they were able to use algebra notation to represent variables and relationships between quantities.

In a study on different notations, Brizuela and Earnest (2008) describe interactions of eight groups of three 4th graders on one particular problem involving a "best deal." Students are moved through four stages of representation of the two deals they compare. First, they are asked to give verbal reactions to the problem and are given chips to help discuss and justify their thinking. Then the groups are asked to represent their thoughts on paper. In response, one student creates a number line that succinctly represents for what values one deal is better than the other. The groups are then asked to make tables and finally asked to graph each of the two deals on a coordinate plane. Throughout the entire interview, students are refining and defining their thinking about the problem. As students are required to use different modes of representation, their mathematical understanding is heightened and the students are provided with "opportunities to infer, confront, and refine ideas. The dynamic relationship among multiple representational systems pushes mathematical thinking to enhance one's overall understanding" (p. 274).

Using a physical model to represent algebraic concepts, Izsak's (2004) analysis of 12 pairs of eighth grade students shows that the students were able to create and use criteria for comparing linear algebraic functions by "drawing on their understanding of physical and numerical patterns, conceptual schemata, and symbol templates" (p. 116-117). He found that "[s]tudents use patterns to recognize situations that can be modeled by algebraic representations" (p. 117). Providing students with meaningful algebraic representations allows them to navigate this initial algebra study with greater independence and understanding.

In a study of pre-algebra classes throughout a school district in California, Brenner et al. (1997) found that "being able to solve the problem and being able to make an appropriate representation were related" (p. 678). With a pre- and post-test and a 20 day intervention for the intervention group, they found that a program rich with the use of language, tables, graphs and algebraic notation created students who were better able to work with linear word problems compared to a group given more traditional algebra instruction. While the intervention group did not score as well on the straight algebraic equation solving problems as the control group, this was to be expected as that was not the focus of their curriculum. "Symbol manipulation skills and word-problem representation skills are cognitive prerequisites for success in algebra, yet traditional instruction may focus on symbol manipulation skills at the expense of representation skills" (Brenner, et al., 1987, p. 684). In a subsequent analysis of the data, Brenner et al. (1997) found that, while their intervention curriculum used a lot more language than the traditional curriculum, and even with the increase in language being used, the Spanish-speaking (English

Language Learner) students in their population became better problem solvers over the course of the intervention than those who received a traditional study of linear equations.

A longitudinal observation of a pair of lower achieving students studying algebra over three years, within a functional approach (Yerushalmy, 2001), specifically "Visual-Math" (CET, 1995), shows the benefit of the approach. Across three interviews of the same pair of students, working a similar problem from year to year, Yerushalmy shows their progress in using strategies that students usually only build over years. In the first interview, the students relied heavily on the only tools they had at their disposal: numeric models. In the second interview, the students employed work with graphs and tables as connected representations. And in the final interview, they used more symbolic representations in the form of sketches and expressions. Over time the students developed more sophisticated problem solving strategies through exposure to thought processes that would not have been introduced in a more traditional algebra curriculum.

Placing the concept of function as central to the discussion of algebra allows connections between algebra topics: "Other constructs in the various secondary school mathematics subjects such as equations, inequalities, identities, relations, formulae, etc. can all be understood in terms of the fundamental object, i.e. the function" (Schwartz & Yerushalmy, 2003, p. 3). A study of functions, relationships and explicit opportunities to work with different representations is the foundation on which further algebraic study and details of algebraic manipulation can be built.

A Functional Approach to Algebra and LD Students: Why?

A functional approach to algebra holds the most promise in helping LD students successfully gain access into the world of algebra. A multi-representational approach that creates meaningful conceptual connections could help these students build a strong conceptual foundation on which they will be able to place all future study of algebra. LD students' arithmetic and algebraic symbol work is often fraught with errors, not necessarily due to comprehension, but to the barriers that their LD creates in terms of limitations of working memory, grapho-motor difficulties, misinterpretation of signs, and difficulty with rote arithmetic learning. A functional approach to algebra would not limit the view of algebra to the alpha-numeric representation, which is typically a most difficult area for these students.

Additionally, having a multi-dimensional view of the subject allows for a greater number of access points for the different learning styles inherent to the LD population. Available studies seem to focus on specific successes or challenges students face with specific interventions based on narrow views of the curriculum. They are connected in saying that representing algebra in more than just its alpha-numeric form helps students meet with the most success. The argument here is that a functional approach that puts the graphical representation either concurrently or earlier than the symbol manipulation integrates a useful representation on which all future algebraic study can be explored.

Brenner et al. (1997) found that a multi-representational approach to algebra helped their students become better problem solvers, including students for whom English is a second language. A functional approach is even accessible to young children (see studies in Kaput, Carraher, & Blanton, 2008). Chazan (2000) found that a functional approach to algebra worked with a low income population. Yerushalmy (2006) shows that low performing students could learn to work with functions in a computer graphing environment. While some students will learn despite the way they are taught, for the population of difficult-to-teach students, studies about students' interactions with a functional approach to algebra seem to give hope for these struggling populations.

Summary

Instead of teaching a traditional curriculum based in abstract symbol manipulation with the concomitant challenges of understanding the symbols, a functional approach to algebra allows students to view algebra from different perspectives, using multiple kinds of representations, instead of focusing solely on alpha-numeric manipulation to solve equations.

While students may learn to parrot procedures for alpha-numeric manipulation within the context of the traditional study of Algebra 1, that is often the extent of their understanding. In my experience, students then view *learning to graph* with the same procedural approach and they have begun to view math as a series of unrelated procedures which they need to memorize in order to become proficient. This does not allow for intuition, conjecturing, and problem solving, nor does it allow those students with arithmetic difficulties to develop a conceptual understanding of algebra.

As an introduction to algebra, a functional approach offers a wider lens to view the subject of algebra, leaving open the possibility of functions other than just linear and quadratic functions. While a functional approach to algebra does not necessarily imply that students will become proficient at manipulating algebraic symbols as they would in a more traditional study of algebra, it does mean that students would see the world of algebra as broader. They might not be so eager to be “finished with algebra” because they would see that mathematics comes as a result of analysis and representations of relationships, not solely as procedures to solve isolated problems. This opens the door to future algebraic study, placing algebra in a larger context, exploring the idea that there isn't an end, but that mathematics is a growing field of study, one which allows for creativity and imaginative thought as well as perspective.

For the LD population, approaching algebra from an angle that is not computation and rigid notation oriented is very helpful. An introduction to algebra that includes a visual element as well as engaging thinking gives these students a framework on which to include the more procedural elements of algebra. Starting Algebra 1 with the study of relationships and continuing with a study of functions, before layering in the alpha-numeric representation for solving equations, has the potential to allow more opportunities for teachers and students to learn how to engage in algebraic conversation that is more than the procedures for manipulating symbols.

Chapter 3: Intervention Curriculum

This chapter describes the new Algebra 1 curriculum. This new curriculum added two chapters to the Algebra 1 course and changed the sequence of the course topics. The four individual teachers preparing to teach Algebra 1 at the time of Study 1 were given the choice to either adopt the new curriculum or continue with the curriculum they had been previously teaching. They unanimously chose to adopt the new curriculum. Since its inception, the sequence of the curriculum has changed very little and the current Algebra 1 teachers are pleased with the conversations occurring in their classroom, as well as with the learning achievements of their students. We will first describe the changes in the course design. After that we will provide an overview with specific examples from the two units added to the start of the course.

Course Changes

In a traditional study of Algebra 1, the alpha-numeric manipulation is introduced first and the graphical representation of algebra last. The intervention curriculum reverses this order, introducing the graphical representation first (see Figure 3) by the addition of two new units (Unit 2 *Algebra as Relationships* and Unit 3 *Functions*) to the first quarter of the Algebra 1 class.

The insertion of these two new units causes a ripple effect on the entire curriculum. Students gain facility working with the graphical representation and that is utilized in each subsequent chapter by first looking at the graphical representation of the concept and then exploring the alpha-numeric manipulation of symbols as a solution strategy.

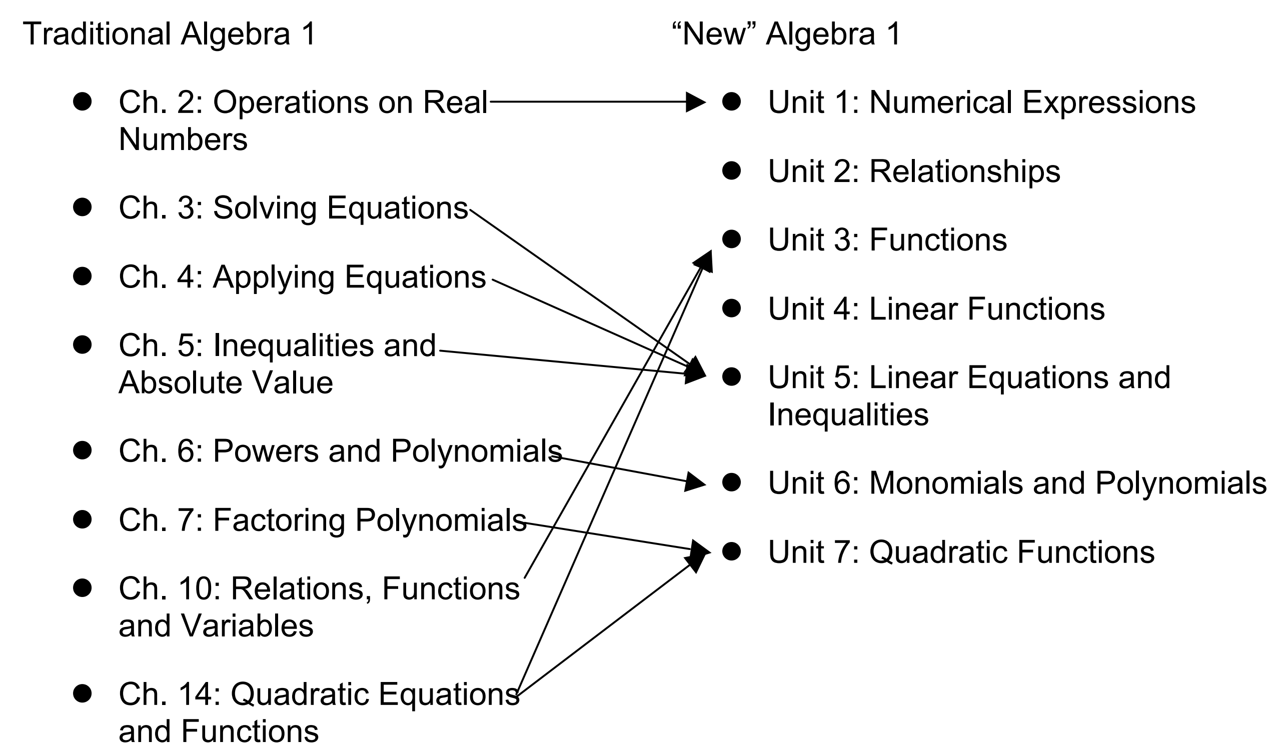


Figure 3: Comparison of traditional curriculum and new Algebra 1 curriculum

The unit on relationships (Unit 2) did not previously exist within the curriculum and when adopted was run concurrently with a review of arithmetic skills. As students are working on the needed review of pre-algebra skills, they are also introduced to the global concept of algebra as the study or relationships, giving them the opportunity to explore bivariate relationships.

The students’ work with the graphical representation of relationships (Unit 2) and their work with arithmetic skills (Unit 1) is merged in Unit 3 with a study of five different functions (linear, quadratic, exponential, square root, absolute value). In this unit the skills of working with functions and their alpha-numeric, tabular and graphical representations is explicitly taught. This requires using skills from the previous two units.

After this more global introduction to algebra, students turn their attention to the family of linear equations: first graphing linear functions (Unit 4) and then solving linear equations

(Unit 5). Graphing linear functions transfers seamlessly from the work on graphing functions explored in Unit 3. Then, with the background in the graphical representation of lines established, students explore solving linear equations through the graphical representation, followed by the solution of linear equations through the alpha-numeric manipulation of symbols. This order, explore and understanding through the graphical representation followed by alpha-numeric symbol manipulation, is continued for each of the subsequent chapters.

The goal of the adopted sequence is three-fold: (1) to introduce the mathematical content to build a wider understanding of algebra before delving into the specific study of linear and quadratic functions, (2) to explore how these different representations are created, how they interact with each other and (3) to begin to reflect on how these skills might be helpful for solving future problems.

The Unit on Algebra as Relationships

In Unit 2: *Algebra as Relationships*, students and teachers explore different aspects of relationships with a heavy emphasis on their graphical representation and verbal description. Emphasis is given to understanding rate of change and the vocabulary for describing relationships is established.

Relationships. As an initial introduction to the unit, students explore a graphical representation of the "Height of the Trees in Mrs. Sauriol's Backyard," which is a first quadrant graph of time and height of a variety of trees (see Figure 4). They are asked to produce statements about the information and discuss as a class how and why they get that information from the graph. Some statements students make compare the trees at a particular instant; others make a comparison about rate of growth. In some classes students made predictions about what the trees might look like currently. Examples of statements by the students are "in 2001, the maple tree is the shortest" and "the pine tree is growing faster than the dogwood." When a student makes a statement they are asked to explain what specifically they are looking at on the graph that prompted that statement.

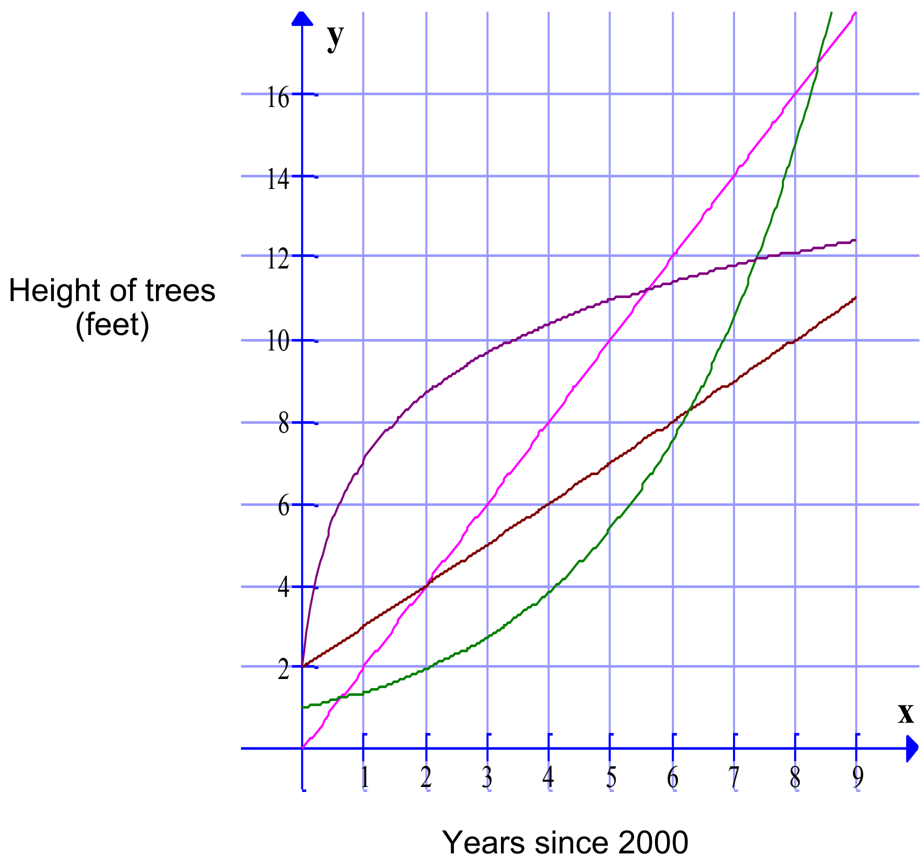


Figure 4: Sample problem - height of trees is Mrs. Sauriol's Backyard

Identifying variables and rate. After an initial introduction with students examining and discussing specific relationships, they are given broader verbally described relationships such as "running a marathon" and asked to identify possible variables in that relationship. They are then asked to complete statements about those relationships, "As <variable> increases, <variable> <increases or decreases>." This allows students to think about which variable is dependent on the other. Since the statements often "sound better" one way as opposed to the other way, this is approached intuitively and discussed openly in the classroom. For instance

“As time increases, distance increases” seems to make more sense than “As distance increases, time increases.” Students begin to differentiate one variable as independent and the other as dependent. This then allows them to set up first quadrant graphs, label the axes and sketch their own relationship graphs. During this time they also discuss their understanding of rate. Most students of this age (14-16 years old) have knowledge about miles per hour and, therefore, recognize distance per time as a reasonable rate. This is also expanded to include other relationships they were working on, using dependent per independent as the rate seems to “make sense” to the students. This is useful later when students began to describe slope, as they are already accustomed to the rate of change relationship (change in y over change in x) without using that specific language.

Rate of change. Students then begin working with story graphs, thinking broadly about whether relationships are positive (increasing), negative (decreasing) or neutral (constant). They also discuss the relative steepness and changing steepness of the curve or line over time. Students engage in discussions related to these graphs and, unlike traditional teachings of algebra, there are many opportunities for different viewpoints, thus allowing students to engage in meaningful dialog to share their perspectives. For instance, when given the prompt “tables in a restaurant” and asked to make a graph of a possible relationship, two students identified the same independent variable (people) and the same dependent variable (number of tables) but their graphs were completely different. One response had a positive constant slope the other response had a negative constant slope (see Figure 5).

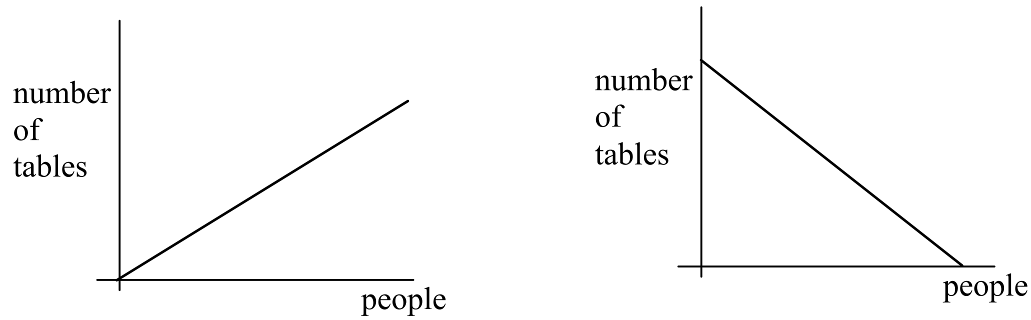


Figure 5: Student responses to describing “tables in a restaurant”

This provided a very rich classroom discussion, as students desired to know who was “right” and the two students had to explain their thought process. The student drawing the positive relationship described that, as the number of people increased, the number of tables needed increased. The other student’s perspective caused him to draw a negative relationship, reasoning that as the people came into the restaurant the number of available tables decreased. Because the question was intentionally vague, it allowed students to do some real reflection and justify their answers. Some other productive conversation ensued concerning better labels for the axes. Perhaps number of tables *available* would be helpful.

Distance versus Time. One of the classroom activities involves a ranger (a distance measuring device) hooked up to a graphing calculator displaying the resulting graphs in real time through an overhead projector (in future classes similar equipment was used with a computer projecting to a screen). Throughout this activity, students explore the very specific relationships between distance and time. This allows them to interact first hand with a distance vs. time graph in real time. They use their bodies and walk toward and away from the ranger to create specific graphs. Students are able to predict how to design their walk (e.g. start close to the ranger, walk fast away, then stop) to obtain a specific graph, and then they are able to test their predictions, gauge their accuracy, and try again if necessary. This specific activity allows students to actually solve problems and kinesthetically feel *rate*.

As found in the detailed analysis of two students' interactions by Nemirovsky, Tierney and Wright (1998), the motion ranger can be a powerful experimenting tool that allows a student to:

> "shift aspects that she perceives as significant in the workings of the tool to the foreground and to relegate idiosyncrasies that must be complied with to the background. In so doing, a sense of logical necessity, what "should" and "should not" happen, becomes part of her emerging perspective (166).

In the case of our classrooms, students were the object that was moving and they could see the immediate effect of their actions. In one particular class, a few students became quite perplexed at how to make a particular graph. After much success with the initial graphs, they became stuck on a graph of varying speed as they tried to match their result with a predetermined shape of a graph. The conversation lasted much longer than anticipated and much need for justification and conversation among the students ensued. This type of interaction, questioning, and exploration is key in developing understanding.

From this experience more story graphs are explored. Throughout the unit, the classes focus on actively using terminology related to describing relationships such as rate (faster and faster, slower and slower, constant), starting position, steepness, independent and dependent variable, positive, negative and constant relationships as it pertains to the comparisons they are making. As noted by Moschkovitch (1996), terms such as "steeper" as it refers to even a linear graph can be confusing, as it can be applied relative to the x or y axis. Thus, it is important that students have an opportunity to use the new terms within different contexts with their peers and through instructional opportunities with their teacher, to solidify their understanding of their meaning.

The classroom dialog allows students to see that looking at a relationship from one perspective yields one type of graph, but looking at it from a different perspective yields a different graph. They recognize that these relationships are not stagnant, thus allowing for some perspective and creativity, an often overlooked ability in mathematics. Most notable to the teachers, is that the discussions in class are rich and interesting, students are exploring, creating and adopting new vocabulary, and they are engaged and listening to each other. The main goal of this unit was to work on analyzing what was being represented in the relationships students were provided with and to be able to communicate that information clearly.

While on one level the class is having these discussions about relationships, they are also reviewing order of operations and operations on integers, decimals and fractions (Unit 1). Even though these are topics typically taught before students attend high school, there is often much confusion and overgeneralizations of the rules they may have learned. Because much of the symbol manipulation that is to come in algebra hinges on these skills, review of these skills is always necessary. Bringing these review topics into this unit also provides a good balance between big picture discussions and procedure oriented calculations. The two major themes balanced each other, providing challenges and opportunities to develop competence for the majority of the students.

The addition of this unit on Relationships accomplishes two things: (1) it sets the tone for the class, recognizing there are many viewpoints and students and teachers need to share and listen to each other and (2) students are exposed to a wide range of concepts with new vocabulary occurring naturally in the context of discussions.

The Unit on Functions

Unit 3, on *Functions*, focuses on aligning the graphical and algebraic representations of different functions. Its first goal is to create a solid foundation and give students experiences on which to base their future discussions about algebra. Another goal is to help students learn how to view algebra discussions through a wider lens. By allowing students to learn how to discuss and look at an algebraic statement in different representations – graphical, algebraic, tabular and descriptive – they are developing abilities they can use in future discussions of the topic. The ability to analyze and verbally describe a graphical or algebraic representation gives students the opportunity to share thoughts and ideas. The need to explicitly work on these aspects is inherent in this population of students.

Explore. For the first activity in this unit of study students are given cards with the alpha-numeric representation of a function, “f(x) = 2x + 3,” on one side and the graphical representation of that function on the other. There are five different functions represented: linear, quadratic, exponential, linear absolute value, square root. Students are asked to sort the cards into five piles (they were not given the categories) looking only at the alpha-numeric representation. This allows them to focus on the salient features of the alpha-numeric representation. Most notable in these interactions is that, given the sorting task, students can distinguish among the various features of the alpha-numeric representation of functions. Some groups start by looking at the numeric values, attempting to put all decimals together and all negative numbers. However, they typically switch strategies independently and move to other more consistent features of the function. Typically groups will pull out all of the square root functions and absolute value functions, then the exponential functions. Some groups have difficulty distinguishing between linear and quadratic functions but, because they know that they are asked to make 5 categories, they eventually separate the categories. The groups are then asked to name the functions. Typical labels they use are “square root,” “absolute value,” “x in the exponent,” “squared.” When asked to give a name to the group of linear functions (without seeing their graphical representation) they often call them the “leftovers” or “plain” implying that there are no distinguishing characteristics for this group of functions. When they turn over the cards to reveal the graph of the functions on the back, they then will name the linear functions “lines.” Students are genuinely surprised that each category has its own graphical features. The relationship between the alpha-numeric and graphical representations is beginning to be established. Typically, in the traditional curriculum, Algebra 1 students are exposed only to lines for the first part of their exploration into algebra – lines, the leftovers, the least interesting of all functions, and ones that students seem to think have no notable characteristics in their algebraic representation and, as noted by Yerushalmy and Schwartz (1993), have characteristics that are not true of other functions.

The goal is to give students a broader view of algebra, beyond the typical Algebra 1 focus on linear and quadratic relationships. With the new curriculum, students are given opportunities to work with five different functions (linear, quadratic, exponential, square root and absolute value). They manipulate and explore these functions, relating back to the vocabulary that they developed in the first unit. Classes explore the relationships between the graphical and alpha-numeric representations through the use of computer graphing software, whose input allows them to type in a function and see immediately the graph of that function. Students try to get the computer to graph a function similar to one sketched. This allows them to explore and play with many different functions without the tediousness of graphing each one by hand.

Notation. Once students have informally interacted with function notation in the card sort and through the computer interface, they are explicitly introduced to function notation and they learn to evaluate functions using that notation. Students are introduced to function notation such as” $f(x) = 3x + 7$, find $f(10)$” as a short hand for “evaluate $3x + 7$ for $x = 10$.” This builds directly from their work with order of operations in the arithmetic chapter.

In addition to responding to notation prompts with arithmetic calculations, students also look at how to get the same information from graphs. This allows very close connections from the graphical and alpha-numeric representations to be established.

Tables and Graphing. Then students work on creating tables, plotting points on the coordinate plane and sketching any of the five different functions presented. The loop, from calculation to table to plotting points, allows students to get feedback on the accuracy of the calculations. Very few LD students become independent with the feedback loop that the graph is providing at this point but, with prompts, most of them can correct errors in their work.

Another important part of this curriculum allows for students to work on precision in plotting on a coordinate plane, a skill that is very disparate among students. In addition, due to the wide range of tracking and graphomotor challenges this LD population faces, the ability to organize and track work on a page, as well as label and plot points, is challenging for some. While the inclusion of how the graphical representation is constructed is tedious, it is an important step for many students.

The connections between the three representations (graph, table, alpha-numeric) are explored and related to the vocabulary developed in Unit 2. In the graphical representation it is relatively easy to determine whether a function is increasing or decreasing and its rate of change (faster and faster, slower and slower, or constant). In the tabular representation, the same can be done but it is more challenging. Helping students begin to view these aspects of the relationship in the table is also explored as a summarizing element to the chapter. These qualities of rate of change are even more difficult to see in the alpha-numeric representation of the function, but making connections between generalized shapes and behaviors of graphs is what is focused on at this point. The goal is to get students to see that each representation offers its own insights into understanding the idea of a function.

This unit on functions provides the balance between the procedural calculations and the exploration of samples of different functions. Assessment and exploring with graphing software help create a connected understanding of the interrelationship between the different representations.

Remaining Algebra Curriculum

The remaining chapters in the Algebra 1 curriculum deal with all the information that is expected in a study of Algebra 1. However, given the background established by the chapters added at the start of the year, the way in which these topics are presented is altered, putting the graphical representation of a concept first and making more connections to the representations of situations.

Linear Functions. Graphing linear functions is studied before solving linear equations. This provides some difficulty as one considers what types of problems are typically presented in a graphing linear functions chapter (i.e. presenting the function as $3x + 4y = 10$ and asking students to put it in the slope-intercept form) because they haven't had the same exposure and may not have the same facility with manipulating the alpha-numeric representation. But what is put at the forefront of this chapter is the idea of modeling linear relationships. Students are able to build a strong understanding of the concept of the y-intercept as a starting point and slope as a rate of change. Students become familiar with the graphical, tabular, alpha-numeric and descriptive aspects of linear functions. Discussions modeled in this unit then naturally lead into the next unit on solving linear equations. What happens when these two lines intersect? At what point is one situation better than another? This is easily answered when the intersection is readily readable on the graph and the answers are in whole numbers, but less so when the answer is a fractional value. At this point students need another way to interact with the same ideas.

Solving Linear Equations. The connection between why we need to solve linear equations is established in the previous unit. Allowing the many different representations of algebra to work with each other and allow students to explore how they work together gives them a very rich and thorough understanding of these foundational algebra topics. As course expectations are including more and more exposure to a wider range of algebra topics in Algebra 1 (Common Core Standards, 2010), the most important thing is that we don't turn students off from math by making it overly complex. The goal of Algebra 1 should be to help students take all of the skills they have acquired in elementary and middle school and integrate them into an understanding of algebra in many different representations.

This suggested approach to algebra allows teachers to remediate arithmetic challenges while building a strong understanding of what algebra is about. It does not build algebra uni-dimensionally on top of a shaky arithmetic understanding. If we want to actualize "algebra for all" we need to recognize that students come to the Algebra 1 classroom with a wide variety of abilities. Helping them integrate and appreciate their study of mathematics, building a strong multi-representational algebra foundation, will do this and it can be done by putting the concept of relationships and functions first.

In the next chapter the method to evaluate the impact of this new Algebra 1 curriculum across the three studies in this dissertation is described.

Chapter 4: General Method

This chapter describes the general method adopted for the three studies. In subsequent chapters, on each of the three studies, more details on the specific aspects pertaining to each study will be described. A description of the classroom intervention activities is described in Chapter 3: Intervention Curriculum.

Participants

All participants in the study are students enrolled in a high school for LD students. Typical class size in this school is between 4 and 8 students. All students have a diagnosed language-based learning disability and their cognitive IQ (based on the WISC) is in the average to above average range. The students' prior experiences with mathematics are quite varied as they come from different schools, different states and even different countries. Within each study, it will be specified whether the student's Algebra 1 programming was given primarily at this school or in a different placement, as this is relevant to the criteria for composition of intervention and control groups. This is also relevant to our assessment of the results, since we are looking at the impact of a particular curriculum piece that was added to the Algebra 1 course of study, which does not currently exist in most Algebra 1 curricula outside of this school.

At this school, students take a year of Algebra 1, followed by a year of Geometry and a year of Algebra 2. Starting with Study 1, students who take Algebra 1 in the school have direct instruction on relationships and functions, as described in Chapter 3: Intervention Curriculum. These two units are included in the first quarter of Algebra 1. Despite this curriculum shift, Algebra 2 classes have continued to teach a more traditional curriculum, which places the alpha-numeric representation first, followed by the graphical representation for linear and then polynomial and quadratic functions.

Each of the three studies compared the performance of an intervention group of students- those who had taken the Algebra 1 course with direct instruction on relationships and functions- to the performance of a control group of students who did not take Algebra 1 at this school and, as such, their instruction with relationships and functions is unknown, but presumed to be minimal. The number of participants in each study is shown in Table 3:

Table 3: Number of Students Participating in each Study

	Intervention	Control
Study 1	40 Algebra 1 students in 10 different classrooms	25 Algebra 2 students
Study 2	31 Algebra 2 students	25 Algebra 2 students
Study 3	36 Algebra 2 students	18 Algebra 2 students

Materials

In each of the three studies, students in each classroom were given a paper and pencil assessment. Each assessment was administered by the classroom teacher as part of the typical activities of the class. In each case, the assessment was delivered during one 45-minute class period. Typically, students finished each assessment in about 30 minutes. The teacher was able to clarify directions and read any of the problems to the students, but did not give feedback on the accuracy of the answers nor any help in the students' choice of answers.

The first assessment (see Appendix B), the "Distance-Time Relationship Assessment," was used in Studies 1 and 2. The second assessment included relationships other than distance and time, the "Relationship Assessment" (see Appendix B), was designed for Study 3.

Distance-Time Relationship Assessment. In Studies 1 and 2, students were given a sheet of paper with the eight distance-time relationship graphs shown in Figure 6 and the following six verbal descriptions:

1. The train left the station and moved away from the station at a constant rate.
2. The train was stopped on the tracks
3. The train was moving faster and faster away from the station
4. The train was coming into the station at a constant rate.
5. The train was coming into the station at a constant rate and then stopped.
6. The train left the station and then suddenly stopped.

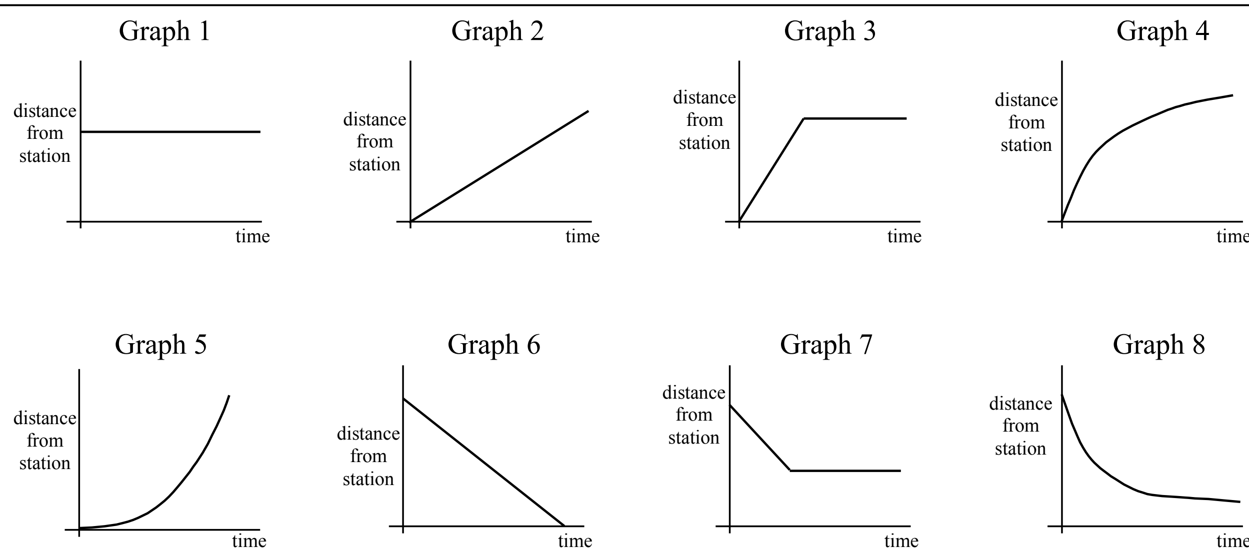


Figure 6: Graphs presented in the Distance-Time Relationship Assessment

For the first part of the assessment, students were asked to cut out the eight graphs and place six of them under the description they judged it best corresponded to. For each of the graphs the x-axis was labeled "time" and the y-axis was labeled "distance from station." Students were encouraged to first lay out and choose all the graphs before taping them down. After matching graphs to the six scenarios, two graphs were left over.

For the second part of the assessment, they were asked to take the two remaining graphs and provide their own description of the relationships represented in each of them.

The third part of the assessment involved students sketching their own graphs for the following written prompts:

1. A train left the station at a steady rate. It stopped suddenly for 2 minutes and then resumed at the same steady rate.
2. A train was about 2 miles from the station when it began to move away from the station at a constant speed.

In the design of the assessment, two important aspects were considered. The first was to allow for judging whether or not participants were choosing the correct graph for a given verbal description. The second was that each scenario had a "target graph answer." This target answer represented a particular scenario and could not represent all the features of other scenarios. For example, in scenario 2, "the train was stopped on the tracks," there are three graphs that actually represent no displacement for at least some of the time (Graphs 1, 3, and 7). Should students choose Graph 3 or Graph 7, when the train would be shown as stopped for only part of the time, that graph would not be available when the appropriate scenario for it was presented.

Relationship Assessment. In Study 3, students were given a sheet of paper with the seven graphs shown in Figure 7 and the following verbal descriptions. Since the variables in each of the verbal descriptions are unique, each question had a labeled x- and y-axis that was related to the problem.

1. There are a dozen eggs in each package. As the number of packages increases, the total number of eggs increases at a constant rate (x-axis: number of packages, y-axis: total number of eggs).

2. As the temperature of the water increases, the amount of oxygen in the water decreases slower and slower (x-axis: temperature, y-axis: amount of oxygen).
3. As more customers entered the restaurant, the number of available chairs decreased at a constant rate (x-axis: customers, y-axis: number of chairs available).
4. The number of lilipads on the pond increases faster and faster each day (x-axis: days, y-axis: lilipads).
5. The cost to mail a letter in the US is 49 cents, no matter how far it travels (x-axis: distance, y-axis: cost to mail a letter).
6. As the number of clouds increased, the number of people swimming decreased faster and faster (x-axis: number of clouds, y-axis: number of people swimming).

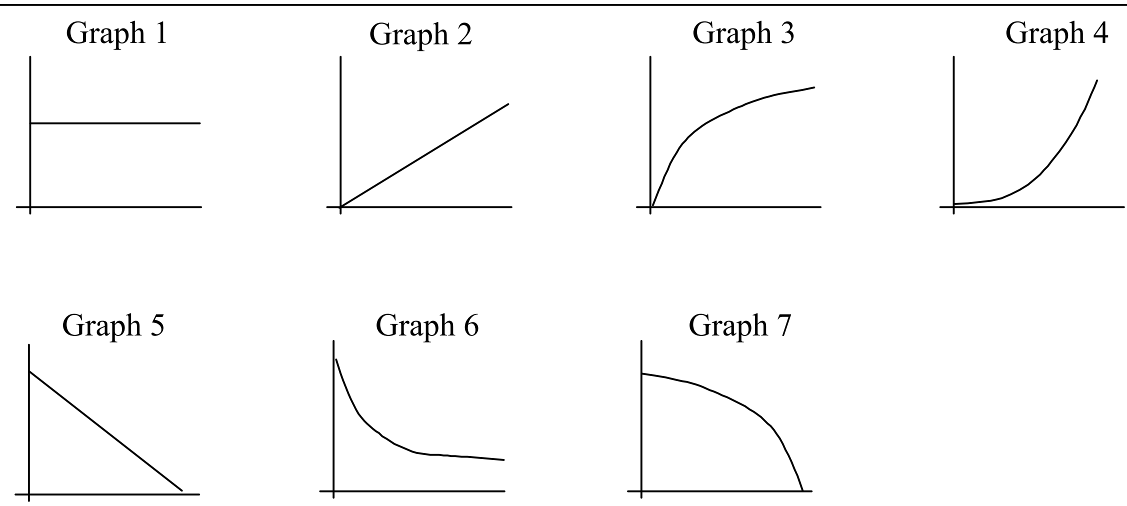


Figure 7: Graphs presented in the relationship assessment

Similar to the other assessment, there were three parts to this assessment. For the first part, students were asked to cut out the seven graphs and place each of them under one of the six descriptions to which they thought it best corresponded. Each of the scenarios had a different x- and y-axis labeled with variables that corresponded to the scenario. Participants were encouraged to lay out and choose all the graphs before taping them down. After matching graphs to the six scenarios, one graph was left over.

For the second part of the assessment, they were asked to look at two graphs (see Figure 8) and generate a description for these graphs. Each of the axes was labeled with variables that were expected to prompt a specific type of response.

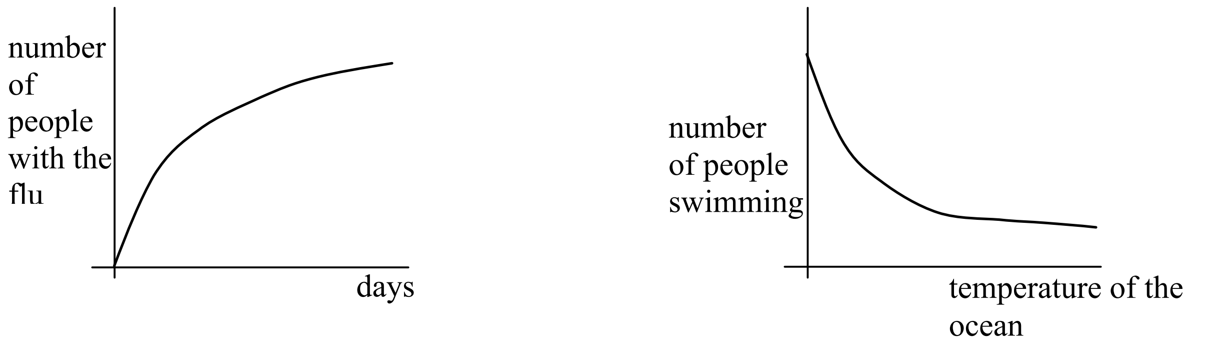


Figure 8: Two graphs for students to describe in the relationship assessment

The third part of the assessment involved students sketching their own graphs for the following written prompts with axes labeled as described in parentheses:

1. The number of lilipads on the pond decreased at a constant rate each day and then one day stopped decreasing and stayed the same (x-axis: days, y-axis: lilipads
2. As the temperature of the water increased, the amount of salt that could be dissolved into the water increased at a steady rate (x-axis: temperature, y-axis: amount of dissolved salt).

Similar to the previous assessment, each scenario in the first part had a targeted graph answer that best suited the scenario. The graphs presented were unique with respect to direction and rate of change, so each description had a targeted selection.

Procedure

In **Study 1,** 40 intervention group students were given the Distance and Time Relationships Assessment, before and immediately after the new Algebra 1 intervention curriculum in which they explicitly studied relationships and functions. While distance and time graphs, as well as many other relational topics, were discussed as part of the instruction, the examples used on the pre- and post-assessment, related to train travels, were not used as part of the teaching in the Algebra 1 classes. In addition, none of the results or scenarios of the pre- and post-assessments were shared or discussed with the students during the lessons. A second group (the control group) was made of 25 students then taking the Algebra 2 course, who had not been introduced to the intervention curriculum. They were given the same Distance and Time Assessment. The goal of Study 1 was to compare the results of the Algebra 1 students in the pre-assessment to those in the post-assessment and to compare their post-assessment results to those of the Algebra 2 students.

In **Study 2** the Distance-Time Relationship Assessment was also used to evaluate 36 intervention students' performances, two years after the intervention, while they were taking their Algebra 2 class. The control group for this study was comprised of 25 students from the same Algebra 2 classes who did not take Algebra 1 at this school. Their exposure to discussions relating to relationships and functions prior to Algebra 2 was unknown. This study was conducted in the spring, as students were working with a unit on polynomials. They had already studied linear functions, including graphing lines, solving linear equations, solving systems of linear equations, as well as function notation and simplifying numerical and algebraic expressions. The approach to this Algebra 2 course is similar to the traditional Algebra 1 course described in the introduction, with a clear emphasis on the alpha-numeric representation.

In **Study 3,** the Relationship Assessment was used to assess 36 intervention and 19 control group students currently studying Algebra 2. As in Study 2, the group was assessed in the spring as they were working with polynomials. The intervention group was made up of students who had taken the modified Algebra 1 curriculum at the school. The control group consisted of students who did not take Algebra 1 at this school, so their exposure to discussions relating to relationships and functions prior to Algebra 2 was unknown.

These three studies were carried out over the course of four years. The first study documents the work done in the first year of the new algebra curriculum. The second study occurred two years later when the students were studying Algebra 2 and had, in the meantime, studied Geometry. The intervention group in the second study was made up of the same intervention group students from Study 1. The third study represented a different cohort of students whose work had not been documented before.

Each of the next three chapters will describe each of the three studies in detail and will present the results for each of them.

Chapter 5: Study 1 – On the short-term effects of the intervention

This first study compares LD students' understanding of linear and non-linear function graphs, before and after they participated in a high school course in Algebra 1 that adopted a functional approach to algebra. Data from this intervention group are compared to a control group of Algebra 2 students who had been exposed to a traditional Algebra 1 course of study.

Method

Generalities as to the types of students and assessment materials were described in the previous General Methods Chapter. Included here are specific aspects of this particular study.

Participants

The intervention group for this study consists of 40 Algebra 1 students distributed over ten classes taught by four different teachers. The teachers' years of experience varied. One class was taught by a teacher with 16 years of experience, six of the classes by two different teachers with three years of teaching experience, and three classes by a teacher with two years of teaching experience. All four teachers had taught a traditional Algebra 1 course prior to this year.

A control group of 25 Algebra 2 students at the same school was used to compare the post-test results. These students have all completed a year of Algebra 1 and a year of Geometry and were halfway through a year of Algebra 2. They had been taught using a more traditional approach and, up to that point this school year, had covered the topics of simplifying expressions, solving linear equations and inequalities, and graphing linear functions. The functional aspects of graphing had not been emphasized with this group.

Assessment and Intervention Procedure

Students in the intervention group were given the assessment as a pre-test in the first quarter of the school year, prior to their study of functions in the Algebra 1 modified course.

After the pre-test was given, students were engaged in a variety of activities that provided them with opportunities to explore and engage in working with linear and non-linear functions. The length of time it took to complete these lessons was between four and seven weeks. The classes met five days a week for 45 minutes. Of that time, 20-40 minutes, 4-5 days of each week, was spent on these activities. The intervention was broken down into two units as described in detail in the Intervention Curriculum Chapter.

As students finished the second unit, they were given the same assessment as a post-test. The pre- and post-tests were given on different days for each class, depending on the course design created by the teacher. The pre-test delivery ranged between the third and fifth week of school. The post-tests were given between 20 and 40 class days after the pre-test in each of the classes. Because the purpose of this study was to evaluate the students based on the pre- and post-tests, the goal was not to have uniform classes.

In each case, the pre- and post-tests were completed during one class period. Typically, all students completed the assessment in about 30 minutes. The teacher was able to clarify directions and read any of the problems to the students, but did not give feedback as to the accuracy of the answers or any help in the students' decision making about their choice of answers.

The same assessment was given to students in the Algebra 2 classes after they had finished their midyear exam. This was the most convenient and least disruptive time to conduct this assessment. Students were given 30 minutes to complete this test, the same time typically needed by the Algebra 1 students. All students were able to finish the test in that time frame.

Results

As described in the Generalized Methods Chapter, in the pre- and post-tests, students were given a sheet of paper with the eight relationship graphs shown in Figure 6. They were asked to cut up the graphs so that they could be taped to a matching description. Each of the graphs was labeled “time” along the x-axis and “distance from station” along the y-axis. During the first part of the test, students were asked to match each graph to a description. For the second part of the test, students were asked to take the two remaining graphs and provide their own description of the relationships represented in each graph.

For the first part of the assessment, we will compare the pre- and post- test results of the intervention group of students. This will be followed by a comparison between the post-test results of the Algebra 1 students to those of the control group of Algebra 2 students. Likewise, the analysis of answers for the second part of the test will focus on comparisons between the descriptions of graphs provided by the intervention group of students before and after the intervention and will be followed by a comparison between the descriptions by the Algebra 1 students to those by the control group of Algebra 2 students.

Part 1 Assessment:

Finding a graph to match a given verbal description

The six scenarios in each of the individual tests were scored as follows: A choice of each of the targeted answers was given two points; a choice of a graph that either represented an appropriate rate or an appropriate direction was given one point; and a choice of a graph that represented neither the appropriate rate or direction was given zero points. The total number of possible points for each student in the first part of the assessment was therefore 12 points.

Table 4 shows the average number of points by students in each group on the first part of the assessment. The table reveals a great improvement on the intervention group’s average scores from the pre-test to the post-test, while the control group of Algebra 2 students shows an average that is only slightly higher than that of the intervention group's pre-test.

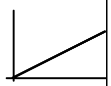
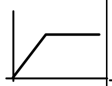
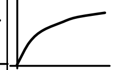
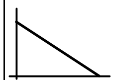
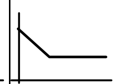
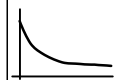
Table 4: Study 1, Part 1: Average Scores for Algebra 1 Pre- and Post-test and Algebra 2 Students

Group	Average Score (maximum 12)
Algebra 1 Pre-test (intervention)	5.5
Algebra 1 Post-test (intervention)	11.1
Algebra 2 Test (control)	6.8

Note: Individual Student Scores can be found in Appendix C

The difference between the intervention group's pre- and post-test scores was significant ($t = 8.78$, $p < 0.0001$), as was the difference between the intervention group's post-test scores and those of the control group ($t = 7.08$, $p < 0.0001$). Surprisingly, the difference between the intervention group's pre-test scores and those of the control group were not significant ($t = 1.38$, $p = 0.1721$). It would be expected that students with more mathematics experience and more exposure to graphing would have better skills for interpreting relationship graphs. Table 5 shows the percentage in each of the intervention and control group choices of graphs for each one of the scenarios that verbally described a relationship.

Table 5: Study 1, Part 1: Algebra 1 Student Pre- and Post-test and Algebra 2 Student Answers to First 6 Scenarios

		Graph 1 	Graph 2 	Graph 3 	Graph 4 	Graph 5 	Graph 6 	Graph 7 	Graph 8 
Scenario 1 The train left the station and moved away from the station at a constant rate.	pre-test	40%	**35%**	7.5%	7.5%	5%	2.5%	2.5%	
	post-test	2.5%	**95%**	2.5%					
	Algebra 2	20%	**40%**	12%	4%	8%	8%	8%	
Scenario 2 The train was stopped on the tracks	pre-test	**37.5%**	17.5%	2.5%	2.5%	15%	17.5%	7.5%	
	post-test	**90%**					2.5%	7.5%	
	Algebra 2	**64%**	8%	4%	12%	12%			
Scenario 3 The train was moving faster and faster away from the station	pre-test		30%	5%	32.5%	**25%**	2.5%	5%	
	post-test				7.5%	**92.5%**			
	Algebra 2	24%	12%	13%	4%	**8%**			
Scenario 4 The train was coming into the station at a constant rate.	pre-test	7.5%	12.5%	12.5%	2.5%	7.5%	**40%**	10%	7.5%
	post-test		5%				**87.5%**		7.5%
	Algebra 2	8%	16%	8%	20%	28%	**4%**	16%	
Scenario 5 The train was coming into the station at a constant rate and then stopped.	pre-test	10%	7.5%	17.5%	2.5%	2.5%	15%	**32.5%**	12.5%
	post-test			5%			5%	**82.5%**	7.5%
	Algebra 2	8%	12%	12%	4%	8%	44%	**12%**	
Scenario 6 The train left the station and then suddenly stopped.	pre-test	2.5%	5%	**37.5%**		2.5%	7.5%	35%	10%
	post-test			**92.5%**				5%	2.5%
	Algebra 2		4%	**64%**	4%	4%	8%	16%	

Note: Percentages in bold are those for correct answers in terms of both rate and directionality.

Intervention Group pre- and post-test comparison

Inspection of the results for the intervention group reveals a striking difference between the pre- and post-test answers in what concerns the spread of the data. For every single question, the range of answers became more concentrated in the post-test, with a much higher percentage of students choosing the targeted answer and the choice of the remaining untargeted answers limited to just one or two other options. By comparison, in the pre-test, the answers were spread among most of the graphs. This focus on the target answers reveals a dramatic shift in students' understanding of how the graphs represented the events described in the verbal statements.

On the pre-test, in response to Scenario 1, only 16 students (40%) chose the targeted answer. There are two important parts to Scenario 1 to which students should have responded to. The first is that the train left from the station. Twenty-two students (55%) chose a graph depicting a relationship that began at the station. The second important part of the scenario is "constant rate." Thirty-five students (88%) chose graphs depicting relationships in which the rate was a constant value for either part or the entire graph. There was only one option for a graph that did not respond to either of these features. This graph was not chosen by any of the students in the pre-test. On the post-test, in response to Scenario 1, 38 students (95%) chose the targeted answer. Thirty-nine (98%) chose a graph that had the train leaving from the station and all 40 students (100%) chose a graph in which the rate was a constant value for either part or all of the graph.

The salient feature of Scenario 2 is the concept of being stopped on the tracks. On the pretest, 15 students (38%) chose the targeted answer. But 32 students (80%) chose a relationship that could be interpreted as stopped at some point on the tracks. In the post-assessment, 36 students (90%) chose the targeted answer and all 40 students (100%) chose a relationship that looks as if the train is stopped on the tracks at some point.

Scenario 3 was the first to introduce the idea of a varying rate. Here, only 10 students (25%) chose the target graph in the pre-test. There are two important aspects of this problem: the “train moving away from the station” which, on the pretest, 37 students (93%) recognized by their choices; and the “faster and faster” type of rate, which only the 10 students (25%) choosing the target answer identified. Only three students (8%) chose a graph with neither of those features in the pre-test. After the intervention, students appeared to be more familiar with the difference between constant and variable rate of change as evidenced by 40 students (100%) choosing a graph in which the rate of change was not constant and the train was depicted leaving the station. However, 3 students (8%) chose an incorrect varying rate leaving 37 students (93%) that did not chose the target answer.

Scenario 4 was the first scenario in which the train was coming toward the station. The answers on the pre-test were again very varied. The two important features of the description were constant rate and direction of the train. In the pre-test, the target answer was chosen by 16 students (40%). Twenty-three students (58%) chose a relationship that represented a train moving toward the station and 33 students (83%) chose a relationship that showed a constant rate. Thirty-six students (90%) chose an answer that responded accurately to at least one of the features of the description. On the post-test, 35 students (88%) chose the target answer, 37 students (93%) chose an answer in which there was a constant rate, and 38 students (95%) chose an answer that depicts the train coming into the station. On the post-test, all 40 students (100%) responded accurately to at least one of the features of the description.

Scenario 5 repeats the idea of the train being stopped on the tracks. On the pre-test, the target answer was chosen by 16 students (40%) while 30 students (75%) chose graphs in which the train was stopped on the tracks for the whole or for part of the trip and 22 students (58%) chose graphs that represented the train coming into the station. On the post-test, 33 students (83%) chose the target answer, all of the students (100%) responded to the idea of the train coming into the station, and 35 students (88%) chose a graph that represents the train stopped on the tracks for all or part of the journey.

In response to Scenario 6, on the pre-test, 33 students (83%) chose a scenario in which the train was stopped on the tracks. Of those 33 students, only 15 students (38%) chose the target answer that included both a portion of the graph representing the train stopped on the tracks and a portion of the graph that represented the train moving away from the station. On the post-test, 37 students (93%) students chose the target answer that included both features of the graph and 39 students (98%) chose a graph that showed the train stopped on the tracks.

The data clearly show that, after receiving instruction of a functional approach to algebra, students of this age, level, and ability can understand and interpret relationship graphs. Prior to the direct instruction and work with these types of graphs and descriptions, students did not fully understand their meaning. After instruction, students understood the salient features of the graphs better, demonstrated by the fact that nearly all students were correct in either part or all of the pairing of the description to each graph.

Intervention vs. Control group comparison

Comparing the results of the Algebra 1 students to the results of the Algebra 2 students (who did not receive the same type of instruction) it is important to note that the answers from the students in the Algebra 2 class more closely match the results of those obtained in the *pre-*

test of Algebra 1 students. As already mentioned, Algebra 2 students showed no significant differences in comparison to the pre-test of the students in Algebra 1. However, comparison of the students in the Algebra 2 test with the post-test Algebra 1 grouping showed extremely significant difference between the two groups with the students of the Algebra 1 group who received the intervention far outperforming those who did not receive intervention despite their years of additional study.

The timing of the test for Algebra 2 students was not at the start of the year, as was the case for Algebra 1 students. The test came at a time when most of the Algebra 2 students had finished a unit on graphing linear equations taught in a very traditional way and had prepared for their mid-year exam. The fact that the students in Algebra 2 were more similar to the pre-test students from Algebra 1 indicates that, despite having successfully completed a course in Algebra 1 and having just finished a unit on graphing linear functions in their Algebra 2 class, they still had not attained an understanding of functional relationships equal that which comes after direct instruction on the topic using a functional approach to algebra. If working from a functional approach to Algebra is important, from these results it seems that students need to directly interact with this idea from the start. Students will not necessarily intuit the functional relationships that are inherent in the graphical representation by studying algebraic manipulations as done in traditional Algebra 1 and Algebra 2 courses.

Part 2: Producing a verbal description for a given graph

After completing the placement of the graphs, students had two graphs remaining. For the second part of the assessment participants were to tape these graphs down and create their own descriptions to explain each of the two graphs. Both directionality and rate were considered in the analysis of the students' generated descriptions. Appendix C shows individual results for the written responses describing the two extra graphs by students in each group. Responses were classified to one of the following nine categories:

- Both descriptors correct
- Speed correct, directionality incorrect or incomplete
- Speed correct, directionality missing
- Directionality correct, speed incorrect or incomplete
- Directionality correct, speed missing
- Speed incorrect or incomplete, directionality missing
- Directionality incorrect, speed missing
- Both descriptions incorrect or incomplete
- Reference to something other than speed or directionality

The frequency of each category of responses for each group is depicted in table 6. Note that a response in which the description of speed was only partially correct was put into a category of speed incorrect or incomplete. This happened only twice and was relevant to students describing graph 3 and graph 7, once in the pre-test and once in the post-test of Algebra 1 students.

What was most notable was the intervention group students' replication of the language that was used in the first part of the assessment. Only a few responses from this group that fall into the "Reference to something other than speed or directionality" category did not use language similar to the prompts from the first part. For a group of students for whom language can be challenging and elusive, use of the language modeled in the first part of the assessment to frame their responses to the second part suggests that language may not be a barrier for them if they are exposed to it in meaningful and predictable ways.

Table 6: Study 1, Part 2: Written Responses Classified into Categories

	Both descriptors correct	Speed correct, directionality incorrect or incomplete	Speed correct, directionality missing	Directionality correct, speed incorrect or incomplete	Directionality correct, speed missing	Speed incorrect, directionality missing	Directionality incorrect, speed missing	Both descriptors incorrect or incomplete	Reference to something other than speed or directionality
Algebra 1 Pre-test	17 (23%)	2 (3%)	7 (9%)	17 (23%)	2 (3%)	4 (5%)	1 (1%)	10 (14%)	14 (19%)
Algebra 1 Post- test	46 (61%)	3 (4%)	8 (11%)	10 (13%)		2 (3%)		1 (1%)	5 (7%)
Algebra 2	4 (11%)	3 (8%)	5 (14%)	4 (11%)	1 (3%)	9 (24%)	2 (5%)	3 (8%)	6 (16%)

Further analysis of the individual responses was done by classifying each response as a level 0, 1, 2, or 3. Examples of each type of response level appear in Table 7. Level 3 responses accurately and completely described speed and directionality of the train. Level 2 responses accurately and completely described the speed of the train, but description of directionality was either incorrect or missing. Some students used the word “left” as in “left fast then slowed down,” because the word “returned” was never used and because the word “left” was used incorrectly in some instances, this word alone was not interpreted to imply direction. Level 1 responses accurately described directionality of the train but a description of the speed was missing, inaccurate, or partially inaccurate. Level 0 responses did not describe either element in a fully accurate manner. These level 0 responses included responses in which one of the elements of speed or directionality was missing, inaccurate, or partially inaccurate and responses that made reference to something other than speed or direction such as “the train was moving in a curved motion.”

Table 7: Study 1, Part 2: Graph 4 and Example Leveled Responses

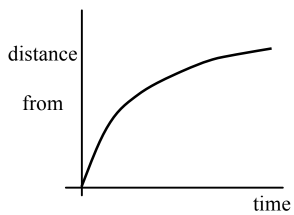
Graph 4	Level responses
	**Level 3:** The train left the station fast then started to slow down
	Level 2: Left fast then slowed down.
	Level 1: The train is leaving faster and faster
	Level 0: The train is coming to the station at a steady rate

Table 8 shows the percentage of responses categorized into each of the four levels. Similar to the first part of the test, the Algebra 1 group's results in the pre-test were somewhat similar to those for the Algebra 2 group. Each of these groups had their highest percentage of written responses scoring at Level 0, while the Algebra 1 post-test group had the highest percentage of written responses scored at Level 3.

Table 8: Study 1, Part 2: Number of Responses at each Level for Intervention and Control Groups

	Level 3	Level 2	Level 1	Level 0
Pre-test	17 (23%)	9 (12%)	19 (26%)	29 (39%)
Post-test	46 (61%)	11 (15%)	10 (13%)	8 (11%)
Algebra 2	4 (11%)	8 (22%)	5 (14%)	20 (54%)

Each individual student's responses were next scored and added giving each student a score between 0 and 6. The individual student's scores are shown in Appendix C and the average for each group is listed in Table 9.

Table 9: Study 1, Part 2: Average Scores for Intervention and Control Groups

Group	Average Score (maximum 6)
Algebra 1 Pre-test (intervention)	2.18
Algebra 1 Post-test (intervention)	4.36
Algebra 2 Test (control)	1.68

As found for the first part of the assessment, in this second part, the difference between the pre- and the post-test scores for the intervention group was significant ($t = 5.1$; $p < .0001$, one-tailed test), the difference between the control group and the intervention group pre-test scores was not significant ($t = 1.002$; $p = .4527$, two tailed test), and the difference between the control group scores and the intervention group post-test scores was significant ($t = 5.628$, $p < 0.001$, two-tailed test).

Discussion

The results of this study were rather positive. Students in Algebra 1, who participated in lessons from the modified curriculum, which included relationships and functions, not only improved their performance on an assessment focused on the interpretation of distance-time relationship graphs, but also performed better than a control group that had received more years of algebra instruction. Specifically, the intervention group could more accurately recognize starting value and varying vs. constant speed in distance vs. time graphs on the post-test than their Algebra 2 peers. In addition to the successful performance of the intervention group, the results indicate that the control group, even though they had received more years of algebra instruction, performed at levels equal to those of the intervention group's pre-test with regard to their ability to describe relationship graphs relating distance and time.

Most surprising of all of the results were the similarities between the pre-test results in Algebra 1 and the results of students in Algebra 2. These results strongly suggest that students do not seem to be given access to ideas related to the interpretation of relationship graphs in the traditional study of alpha-numeric symbol manipulation, even after years of algebra instruction. This lends credence to the belief that students previously saw graphing as something you do, not something that is meaningful. By contrast, the approach evaluated in this study allowed students to interact with the graphical representation in a meaningful way and, as the results show, to become able to understand relevant information from the graphs.

After the intervention, students showed a number of strengths related to their ability to pair verbal descriptive representations to a distance vs. time graph. Students in the new Algebra 1 curriculum did increase their ability to:

- accurately recognize directionality
- accurately recognize varying vs. constant rate of change
- more consistently use language to describe directionality
- more consistently use language to describe varying vs. constant rate of change

As a result of the intervention, students in Algebra 1 increased their understanding and interpretations of linear and nonlinear graphs. Most important of all, this study shows that LD students are capable of a sophisticated view of functions graphs that includes their ability to interpret graphs into meaningful language despite their language-based learning disability.

As noted in the results of the pre-test of the Algebra 1 students and the test given to the Algebra 2 students, the study of rate of change is an elusive idea. However, after the intervention, these intervention group students were able to accurately identify representations that related to varying versus constant rate of change and choose graphs based on that feature.

Another interesting finding of this study was the LD students' ability to work with and to adapt to the language that was presented by the instructors. The majority of students used the language presented in the first part of the assessment as a model for the language they used for the second part. This lends credence to the social constructivist notions of mathematics being navigated by experiences and language. Students need language to interpret what they are seeing. When the language is removed, as in the case of pure algebra alpha-numeric manipulation, that need is not supported. By beginning algebra with the navigation and meaning of these ideas, students gain confidence in their ability to apply meaning to what they are seeing.

Given the results of previous research showing that students have difficulty with interpreting graphs, the results of this study are extremely encouraging, for LD as well for any population of students. This study leads to the conclusion that, with focused attention on this difficulty, students are able to make sense of graphs and are able to more accurately interpret graphs. The next question explored in Studies 2 and 3 concerns the long-term effect of this initial introduction to algebra within a functional approach on students' performance on the same written assessment examined in this chapter (see Study 2) and in a written assessment with questions on relationships between variables other than time and distance (see Study 3).

Chapter 6: Study 2 – Longitudinal Analysis of Data from Study 1

The second study analyzes LD students' understanding of linear and non-linear function graphs, two years after the curriculum intervention in their Algebra 1 courses. These students completed a study of Algebra 1, a study of Geometry, and were three-quarters of the way through a course in Algebra 2. Data from the group who, in Study 1, completed Algebra 1 with the relationships and functions curriculum described in Intervention Curriculum Chapter are compared with students who did not participate in the Algebra 1 program at this school.

Method

The General Methods Chapter provided an overview of the participants and materials in this second study. Specific aspects of this particular study are described next.

Participants

Fifty-six LD students enrolled in an Algebra 2 course at a school for LD students participated in the second study. The intervention group consisted of 31 students who had received Algebra 1 instruction at this school, which began with a study of relationships and functions. The control group consisted of 25 students who had transferred to this school after completing Algebra 1 at a different school; it is unknown whether their study of Algebra 1 included a discussion of relationships and functions. Of the 31 students in the intervention group, 26 were part of Study 1. The other five students had completed Algebra 1 at this school with the modified curriculum, but had not been part of Study 1 because their written assessments were incomplete.

As is typical of the school where the studies took place, students are not necessarily a part of the program for four years. They often are transferred from other schools, as was the case of the 25 students who made up the control group. For 12 control group students this was their first year at this school. The other 13 students took Geometry at this school and this was their second year of instruction at the school.

At the time of the assessment, all of the students were enrolled in an Algebra 2 course. This Algebra 2 course presents a traditional view of algebra. At the point in the year when the students were tested, they were working on the study of polynomials. They had completed studies of simplifying expressions, solving linear equations and inequalities, and graphing linear functions prior to this point. The discussion of relationships other than specific models presented in the textbook and relating to linear modeling was not explicitly emphasized in the curriculum.

Assessment Materials and Procedure

The same written assessment used in Study 1 was administered to the 56 Algebra 2 students that constituted the intervention and control groups. As described before, the assessment involved three parts. In the first part, students were given eight relationship graphs to be matched to six verbal descriptions by cutting and pasting the graphs onto a worksheet. Each of the graphs was labeled “time” along the x-axis and “distance from station” along the y-axis. For the second part, students were to provide a verbal description of two graphs that were not matched to any verbal description in the first part. The third part of the assessment asked students to sketch a graph of their own for each two verbal descriptions.

As in Study 1, students were given a 45 minute class period to complete the assessment. Typically, the students finished within 30 minutes. The assessment was administered by their classroom teacher. If the students requested it, questions could be read aloud and directions could be clarified, but no feedback as to the accuracy of their answers or help in students’ decision making about choice of answers was given.

Results

Similar to Study 1, the results of the intervention and control groups will be compared for the first part of the assessment. Further analysis of individual results for the last two parts will also be included to investigate the patterns of individual student errors to look for consistent patterns of errors.

Part 1: Finding a graph to match a given verbal description.

The six scenarios in the individual tests were scored giving each answer 0, 1, or 2 points for a possible total of 12 points, as was done in Study 1. A choice of each of the targeted answers was given two points, a choice of graphs that either represented an appropriate rate or an appropriate direction was given one point and no points were given for choices that did not represent the appropriate rate nor the direction.

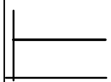
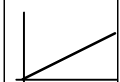
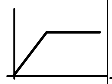
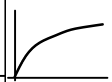
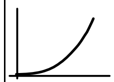
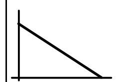
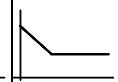
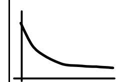
Table 10: Study 2, Part 1: Average Scores for Intervention and Control Groups

Group	Average Score (maximum 12)
Intervention	10.42
Control	7.76

Note: Individual Student Scores can be found in Appendix D

As expected and as was the case for Study 1, the intervention students performed better than the control students. The difference between the intervention and control groups' test scores was statistically significant (Mann-Whitney U = 222, z = 2.72, $p < 0.01$). Table 11 shows the intervention and control groups' choices of graphs for each one of the scenarios that verbally described a relationship.

Table 11: Algebra 1 Student Pre- and Post-test and Algebra 2 Student Answers to First 6 Scenarios

		Graph 1	Graph 2	Graph 3	Graph 4	Graph 5	Graph 6	Graph 7	Graph 8
									
Scenario 1 The train left the station and moved away from the station at a constant rate.	intervention		80.6%		6.5%		12.9%		
	control	28%	60%			4%	4%		4%
Scenario 2 The train was stopped on the tracks	intervention	100%							
	control	60%	8%	16%		4%	4%		8%
Scenario 3 The train was moving faster and faster away from the station	intervention		6.5%		9.6%	77.4%			6.5%
	control		16%		12%	60%	12%		
Scenario 4 The train was coming into the station at a constant rate.	intervention		12.9%			3.3%	74.2%		9.6%
	control						80%	16%	4%
Scenario 5 The train was coming into the station at a constant rate and then stopped.	intervention			12.9%		3.2%		77.4%	6.5%
	control	4%		20%	4%		4%	60%	8%
Scenario 6 The train left the station and then suddenly stopped.	intervention			83.9%				16.1%	
	control	4%	4%	48%		4%	12%	20%	8%

The distribution of answers suggest that the positive effect of the intervention curriculum still holds. Specifically, with the exception of Scenario 4, the percentage of choices of the targeted response by students without exposure to the relations and functions curriculum is lower than those for the intervention group. For intervention students, the positive effects of the modified Algebra 1 curriculum persist after 24 months, as shown by the consistency in their responses and the accuracy over their same class peers who did not receive Algebra 1 instruction at this school. In the intervention group, 19 of the 31 students (61.3%) chose the targeted answer each time. In the control group, only 7 of the 25 students (28%) chose the targeted answer each time. Next the incorrect responses from the remaining 12 students from the intervention group and 18 students from the control group will be further examined.

Part 1: Further Analysis of Errors

As discussed before, for each graph there are two major features to be considered. The first is directionality: if the description is that the train is moving away from the station, the graph should show a positive relationship; if the description is that the train is coming into the station, it should show a negative relationship. The second major feature of the graph is rate, shown in the curvature of the graph: if the description is a constant rate there should be no curvature of the graph; if the description mentions an increasing or decreasing rate, the graph should have the appropriate curvature.

Considering their responses in relation to directionality and rate, students who did not consistently choose the targeted answer fell into one of three other categories. The first category includes students who chose graphs that consistently represented the correct directionality of the train but, at least once, chose a graph that did not show the correct rate. The second category includes students who consistently chose graphs depicting a direction that was the opposite of the targeted direction, but with a curvature that was consistent with the rate given in the verbal statement. Answers to Scenario 3 did not include a graph that would fit into this category (opposite direction, correct rate) and, for that reason, answers to this scenario were omitted when determining if a student would fit into this second category. The third category of responses included those showing wrong answers with no discernible pattern of errors regarding direction or rate. The number and percentage of students from the intervention and control groups in each category is shown in Table 12.

Table 12: Study 2, Part 1: Response Category of Comparison of Intervention and Control Groups

	Intervention	Control
Both direction and rate correct	19 (61.3%)	7 (28%)
Direction correct; rate inconsistent	6 (19.3%)	4 (16%)
Direction wrong; rate correct	4 (12.9%)	0
Inconsistency pattern of errors regarding rate and direction	2 (6.5%)	14 (56%)

Further analysis of the students in each of these three categories will be explored in relation to the remainder of their answers to the assessment.

Direction correct, Rate inconsistent. In teaching students to read a relationship graph, one of the first important things that students become aware of is the direction of the graph. They learn to read the graph from left to right and may develop an understanding of the concept of positive or negative relationships. As students become more proficient, they work on the detail of what the curvature of the graph adds to the discussion of rate of change. Students in this category seem to understand the directionality of the train, moving toward or away from the station, but the concept of constant rate, faster and faster or slower and slower is not well established.

Six students (19.3%) from the intervention group and four students (16%) from the control group fit into this “Direction correct, Rate inconsistent” category. For the first part of the assessment, these students chose a graph that had the correct directionality but not necessarily the correct rate. Their graph choices can be seen in Table 13.

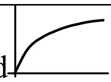
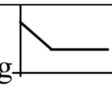
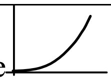
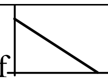
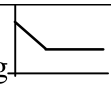
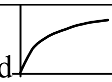
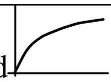
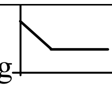
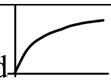
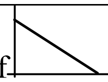
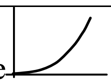
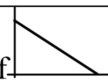
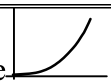
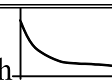
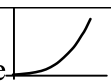
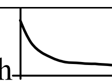
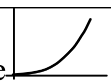
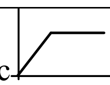
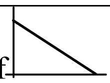
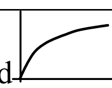
Table 13: Study 2, Part 1: Responses of Intervention and Control Group Students Who Show Choice of Correct Direction But Not Correct Rate

Student Number	**Scenario 1** The train left the station and moved away from the station at a constant rate.	**Scenario 2** The train was stopped on the tracks	**Scenario 3** The train was moving faster and faster away from the station	**Scenario 4** The train was coming into the station at a constant rate.	**Scenario 5** The train was coming into the station at a constant rate and then stopped.	**Scenario 6** The train left the station and then suddenly stopped.
014 Intervention	b	a	d	f	g	c
027 Intervention	b	a	d	h	g	c
028 Intervention	b	a	e	f	h	c
041 Intervention	b	a	e	f	h	c
083 Intervention	b	a	e	h	g	c
020 Intervention	d	a	b	h	g	c
101 Control	b	a	d	f	g	c
102 Control	b	a	d	f	g	c
106 Control	b	a	d	f	g	h
120 Control	b	a	e	g	h	c

Note: Yellow highlighted boxes represent correct direction and incorrect rate; green highlighted boxes represent neither description correct.

The starting points for each of the chosen graphs (highlighted in yellow) corresponded to the starting point in the prompt, but not to the rate described in the statement. In the second part of the assessment, where they were asked to provide their own description of the relationships represented in the two graphs that were not matched to a verbal description in the first part, five of the intervention students and the four control students continued to show accuracy regarding the direction of the train, but not necessarily the rate (one intervention student did not provide a response for the second part of the assessment). Table 14 shows the responses from the six students in the intervention group and four students from the control group who fit in this subcategory.

Table 14: Study 2, Part 2: Responses of Direction Correct, Rate Inconsistent Subcategory from Intervention and Control Groups

014 Intervention	d 	The train moved away from the station slowing down	g 	The train was going toward the station and stopped
027 Intervention	e 	The train started to move slow out of the station, then gained momentum	f 	The train slowed down as it moved closer towards the station
028 Intervention	g 	The train had left at a constant speed then it slowed down	d 	It left the station at a steady/speedy rate
041 Intervention	d 	The train moved away from the station slowing down	g 	The train was going toward the station and stopped
083 Intervention	d 	The train is leaving the station and slowly moving away	f 	The train is coming into the station at a constant speed
020 Intervention	e 	No description given	f 	No description given
101 Control	e 	The train left slow and sped up quickly	h 	The train came in very quickly and suddenly stopped
102 Control	e 	The train quickly left the station	h 	The train slowly came into the station
106 Control	e 	The train left slowly then gained speed	c 	The train left at a constant rate then stopped
120 Control	f 	The train was coming back to the station	d 	The train pulled away from the station and arrived at the second stop

Note: Yellow highlighted boxes represent correct direction and incorrect rate; green highlighted boxes represent neither description correct.

Two of the intervention students chose the correct rate in each response, but the other three provided a wrong description regarding the rate depicted in the graph. Two of the control group students interpreted the graphs accurately with respect to rate and direction, while the other two accurately described direction but not rate.

The third part of the assessment asked students to draw graphs of the movement of the train in response to verbal descriptions of two trips. Given the two prompts, each of the six intervention students drew graphs with the correct direction, with a train leaving the station. Five of the six students provided a sketch to the first verbal description (Question 9) similar to the one shown in Figure 9, where direction, rate and starting point (y-intercept) of the sketches accurately represented the description. The sixth student's sketch (student 028) correctly depicted directionality but not rate. All four students in the control group drew a graph similar to the one below for question 9, with correct direction and rate.

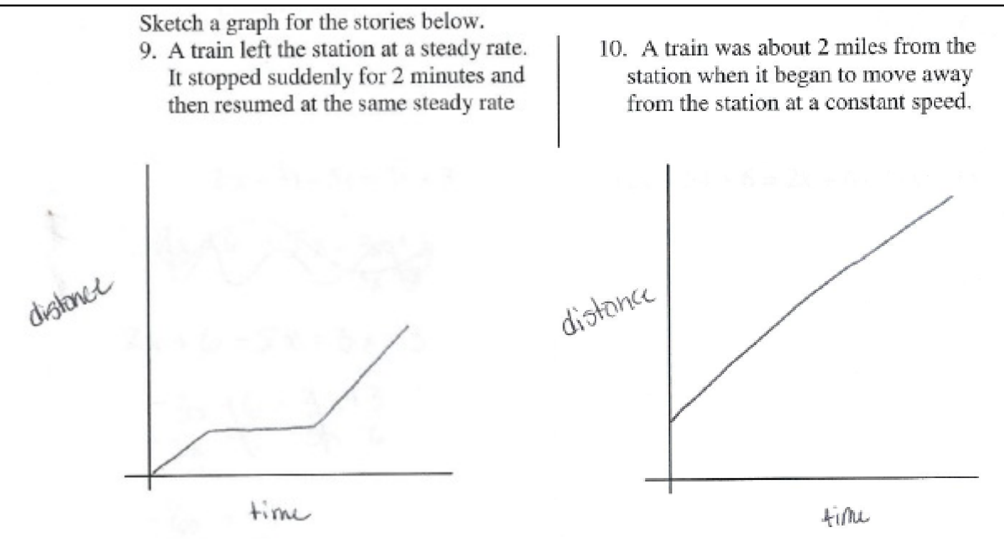


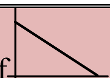
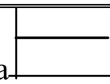
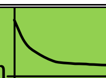
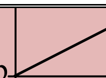
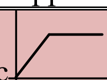
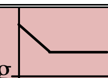
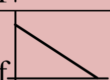
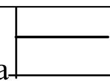
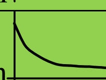
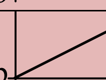
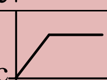
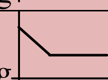
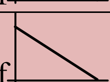
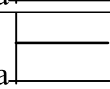
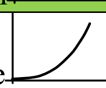
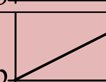
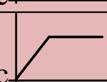
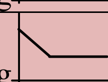
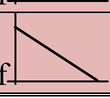
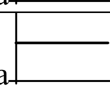
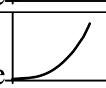
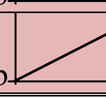
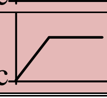
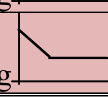
Figure 9: Study 2, Part 3: Sample student responses to third part of assessment for subcategory of students who consistently represent direction but not rate

In response to the second verbal description (Question 10), all six of the students in the intervention group sketched graphs with accurate direction and starting point. Three of these students provided sketches similar to the one for Question 10 in Figure 9, showing accurate rate. The remaining three graphs (students 028, 041, 083) had different concavity showing inconsistencies with respect to rate. Three of the four students in the control group drew a line with a positive slope, but their y-intercepts were in different places. Student 101 depicted the starting place at the origin. Student 102 started the line on a positive point on the x-axis. Student 106 had a positive y-intercept similar to the one shown in Figure 9. The remaining control group student (120) drew a graph that started with a negative slope, then a flat period and then a positive slope.

With few exceptions, the students continued to interpret graphs accurately with respect to direction, but did not show full understanding of how to represent rate, as evidenced by their inconsistent choices, descriptions and drawings of the curvature of the graph.

Direction Incorrect, Rate Accurate. It is not uncommon for LD students to have difficulty with directionality in their work, making many of the conversations relating to graphing very challenging for them. Four of the students in the intervention group (12.9%) showed correct choices for rate of change but consistently incorrect choices for direction. No students in the control group fit into this category. Their responses for the first part of the assessment are shown in Table 10.

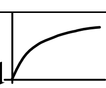
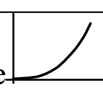
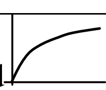
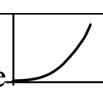
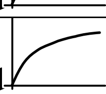
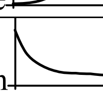
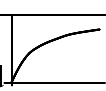
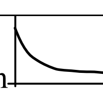
Table 15: Study 2, Part 1: Responses of Intervention Group Students Who Show answers for rate but Consistently incorrect answer for Direction

Student Number	**Scenario 1** The train left the station and moved away from the station at a constant rate.	**Scenario 2** The train was stopped on the tracks	**Scenario 3** The train was moving faster and faster away from the station	**Scenario 4** The train was coming into the station at a constant rate.	**Scenario 5** The train was coming into the station at a constant rate and then stopped.	**Scenario 6** The train left the station and then suddenly stopped.
007 Intervention	f 	a 	h 	b 	c 	g 
015 Intervention	f 	a 	h 	b 	c 	g 
002 Intervention	f 	a 	e 	b 	c 	g 
054 Intervention	f 	a 	e 	b 	c 	g 

Note: Pink highlighted boxes represent incorrect direction and correct rate; green highlighted boxes represent neither description correct.

Their responses to the second part of the assessment, depicted in Table 16, shows the same pattern, with correct answers for rate but not for direction in every case..

Table 16: Study 2, Part 2: Responses of Intervention Group Students Who Show Consistent Accuracy with respect to rate and Consistently Opposite Direction

007 Intervention	d 	The train started to slow down as it got closer to the station	e 	The train went faster and faster towards the station
015 Intervention	d 	The train slowed down as it approached the station	e 	The train sped up as it approached the station
002 Intervention	d 	The train was heading towards the station while slowly decreasing speed	h 	The train left the station at a fast pace and then slowed down
054 Intervention	d 	The train was coming into the station but started to lose gas	h 	When the train was coming to its stop, it slowed down

The third part of the assessment revealed similar results. Three of the four students responded to Question 9 with drawings similar to the first one in Figure 10. The fourth student (002) sketched a graph somewhat similar to this one, but instead of a straight line, drew a line concave down for each segment. These students also produced similar but different graphs for Question 10. One student drew the graph in Figure 10 for Question 10, with a plateau at the start. Another just showed the decreasing segment hanging in mid graph, with nothing connecting it to the y-axis. A third student drew a decreasing slope line directly from a point high on the y-axis, and a fourth student (002) drew an increasing slope line from the origin with an inflection point midway up the line. This student had also triple-traced over her word "distance" on the side of this graph, while retracing words is not uncommon in the LD profile, and can be a form of doodling or thinking, this student did not do this on any other aspect of the assessment, indicating perhaps that she was unsure of what she was doing in relation to the rest of the testing. While these four students represent part of the scatter of the responses in analyzing the first part of the assessment, their responses remain consistent throughout the assessment showing incorrect interpretation of direction but a consistently accurate interpretation of speed.

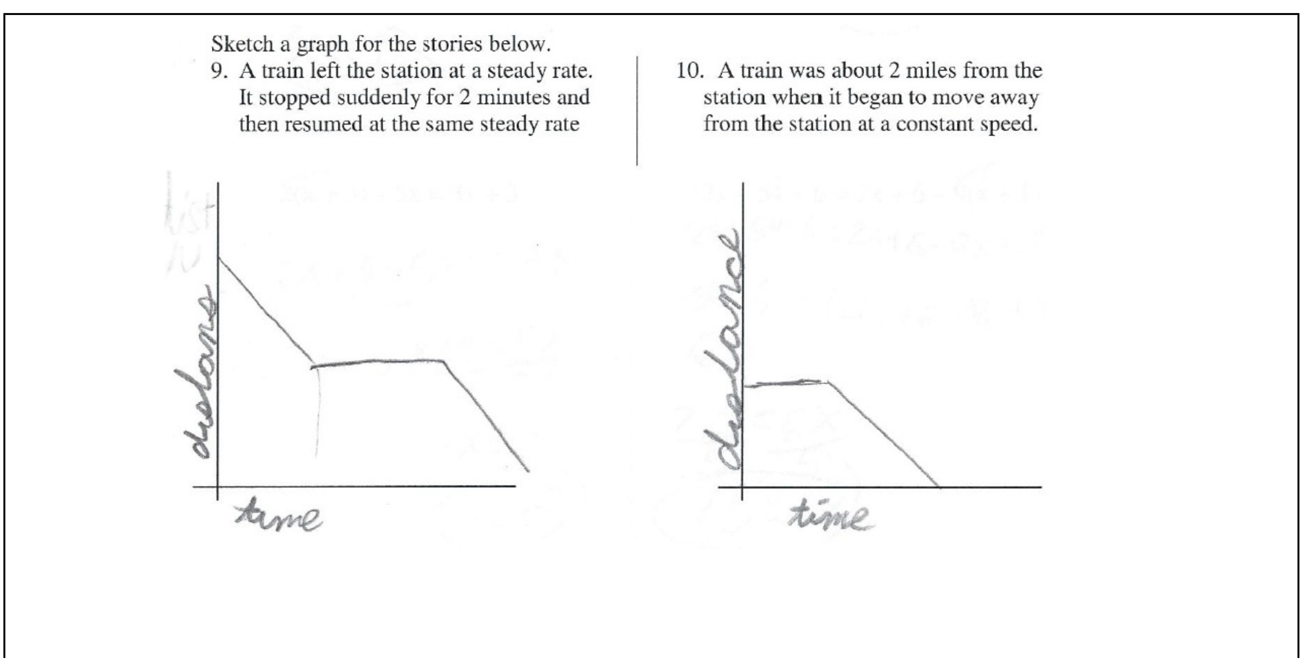


Figure 10: Study 2, Part 3: Sample student response to the third part of the assessment for students with incorrect direction and correct rate

This subcategory of responses appeared throughout the three parts of the assessment. Apparently, these students have some understanding of relationships graphs but struggle with considering the representation of direction. It is interesting to notice that this subcategory of responses only appeared for the intervention group.

The third category of students showed inconsistent patterns of errors regarding rate and direction. In Study 1, a wide variety of responses with no clear pattern of errors was shown by students who had little or no exposure to direct instruction on relationships and functions. These types of responses were evident in the pre-assessment of Algebra 1 students and in the students in the Algebra 2 class (before the new curriculum had been introduced). In this study, only two of the intervention group students (6.8%) and over half of the control group students (14 students, 56%) showed this inconsistent pattern of errors (see Table 17).

Table 17: Study 2, Part 1: Responses of Remaining Intervention and Control Group Students not Choosing Targeted Responses and Showing No Consistent Pattern of Errors

Student Number	Scenario 1 The train left the station and moved away from the station at a constant rate.	Scenario 2 The train was stopped on the tracks	Scenario 3 The train was moving faster and faster away from the station	Scenario 4 The train was coming into the station at a constant rate.	Scenario 5 The train was coming into the station at a constant rate and then stopped.	Scenario 6 The train left the station and then suddenly stopped.
001 Intervention	d	a	b	f	e	g
003 Intervention	b	a	d	e	g	c
123 Control	b	e	f	h	a	g
104 Control	a	c	e	f	g	b
103 Control	b	a	e	g	c	f
125 Control	b	a	e	f	c	g
119 Control	f	a	e	b	g	c
105 Control	b	a	e	c	d	g
118 Control	a	c	e	b	g	f
117 Control	a	c	b	f	g	e
116 Control	h	b	e	d	f	a
122 Control	a	h	f	b	g	c
110 Control	a	f	b	g	c	h
111 Control	a	h	b	g	c	f
114 Control	e	c	b	a	h	g
121 Control	a	b	f	e	c	g

Note: Yellow highlighted boxes represent correct direction and incorrect rate; red highlighted boxes represent direction incorrect, rate correct; green highlighted boxes represent neither description correct; brown highlighted boxes represent an added a piece of information not in the prompt

These kinds of responses with no clear pattern of errors appeared through the remaining two sections of the assessment. When the responses to the other parts of the testing for the two students from the intervention group are analyzed, it is seen that in the second part of the assessment - intervention student 003 described one graph accurately and the other graph inaccurately with respect to both direction and rate. In the third part of the assessment, intervention student 003 produced graphs similar to those in Figure 9, showing an accurate representation of events in her drawing. In the second part of the assessment, intervention student 001 did not describe either rate or direction and simply stated “the train is moving” for both graphs. The third part of the assessment, this student produced graphs that were incomplete or inconsistent with the verbal description.

The remaining fourteen students in the control group (56%) continued to show no clear pattern of errors throughout the remaining parts of the assessment. In addition, some of them in their written descriptions of the graphs included something other than rate and direction. For instance, “the train was gaining speed a lot at the beginning and rising little by little” and “the train was moving at a steady rate, turning at the same time.” These are the types of errors that were noticed in Study 1 with the students who had no experience with relationship and function instruction.

Discussion

The results of this study are as encouraging as those of Study 1, for LD as well as for any population of students. Over 24 months after the intervention, students continued to show a number of strengths over their peers from the control group in terms of the ability to pair graphical and descriptive representations of distance - time relations. Over half of the students not receiving the intervention curriculum could not consistently interpret these graphs. For these students, without direct instruction on how to read and interpret relationship and function graphs, they could not succeed in the assessment. However, instruction given early in the high school career of students can have a lasting impact on their ability to interpret relationship graphs. Study 3 will further examine the impact of the new curriculum on the interpretation and production of graphs with variables other than time and distance.

Chapter 7: Study 3 – Analysis of Relationship Graphs

The third study analyzes LD students' understanding of linear and non-linear function graphs in another sample of intervention and control students, from the same school. Different from Studies 1 and 2, this time the relationships described by the verbal problems and graphs referred to different, less familiar contexts, instead of the time-distance relationships regarding train travel, as was the case in the previous studies. Studies 1 and 2 were extremely encouraging in documenting the differences between students who had received the intervention curriculum. This third study further explores what students in the intervention curriculum gained over their control group peers. This study aims to see if the differences between the intervention and control group still hold for graphs about variables other than time-distance. It will also inform decision making about the Algebra 1 curriculum.

Method

Participants

Fifty-four LD students enrolled in an Algebra 2 course at the same school for LD students participated in the third study. The intervention group consisted of 36 students who had received Algebra 1 instruction at this school, with the modified curriculum, which began with a study of relationships and functions. The control group consisted of 18 students who had transferred to this school after completing Algebra 1 at a different school, so it is unknown whether their study of Algebra 1 included a discussion of relationships and functions.

As is typical of the school where the studies took place, students are not necessarily a part of the program for four years. They often are transferred from other schools, as was the case with the 18 students who made up the control group. For ten students this was their first year at this school. Eight students took Geometry at this school and this was their second year of instruction at this school.

At the time of assessment, all of the students were enrolled in an Algebra 2 course. This Algebra 2 course is a traditional presentation of algebra. At the point in the year when the students were tested, they were working on a study of polynomials. They had completed studies of simplifying expressions, solving linear equations and inequalities, and graphing linear functions prior to this point. The discussion of relationships other than specific models presented in the text relating to linear modeling was not emphasized in the curriculum. However, because a focus on functions and relationships had been adopted by the Algebra 1 group three years before and due to the individualized nature of instruction at this school, some of the Algebra 2 teachers (all three had been at this school for more than 4 years) had begun to include aspects of the Algebra 1 intervention curriculum within their classes. Students who had taken the new Algebra 1 course came to their classes with different terms for describing functions and these teachers adapted and incorporated this vocabulary into their instruction for all students. While this was not necessarily deliberate, the cultural shift of the algebra classes at this school had begun to change and this population was not as removed from the intervention as it had been previously.

Assessment Materials and Procedure

A written assessment, similar in format to Studies 1 and 2 assessments, was administered to the 54 Algebra 2 students. However, in this case, the problems involved a wider variety of scenarios, many of them non-temporal, instead of distance and time problem about a train's displacement. The assessment involved three parts. In the first part, students were given seven relationship graphs to be matched to six verbal descriptions by cutting and pasting the graphs onto a worksheet. Each of the prompts had a labeled x- and y-axis that were unique to each problem. For the second part, students were to provide a verbal description of two graphs with a labeled x- and y-axis. The third part of the assessment asked students to sketch a graph of their own and label the x- and y-axis from a verbal description. In this assessment each question required students to consider a new pair of variables.

The same administration techniques used in Studies 1 and 2 were used here, students were given a 45 minute class period to complete the assessment administered by their classroom teacher. Typically all students finished within 30 minutes. If the student requested it, questions could be read aloud and directions could be clarified, but no feedback as to the accuracy of their answers or help in students' decision making about choice of answers was given.

Results

Part 1: Finding a graph to match a given written description.

The six scenarios in the individual tests were scored giving each answer a zero, one or two points for a possible total of 12 points. A choice of each of the targeted answers was given two points and a choice of other graphs that either represented an appropriate rate or an appropriate direction was given one point. A choice of a graph that did not represent the appropriate rate or direction was given no points. Table 18 shows the average scores for each group.

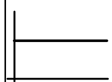
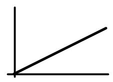
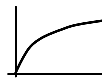
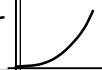
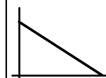
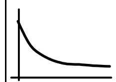
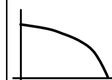
Table 18: Study 3, Part 1: Average Scores for Intervention and Control Groups

Group	Average Score (maximum 12)
Intervention	10.94
Control	10.32

Intervention students performed slightly better than the control students, but the difference between the two groups was not significant (Mann-Whitney $U = 270$, $z = 1.27$, $p > 0.10$). Surprisingly, the scores for the control group were similar to those of the intervention group in the previous two studies where the average of correct answers for the intervention groups were 11.1and 10.42 from Study 1 post-test and Study 2, respectively.

Table 19 shows the intervention and control groups' choices of graphs for each one of the scenarios that verbally described a relationship. In this part of the assessment, 24 students from the intervention group (66.7%) and eight from the control group (44.4%) chose the targeted correct answer for all six items. Thus, while the average scores between the control and intervention group are very similar, the percentage of students from the intervention group choosing the targeted graph each time is greater than the percentage from the control group.

Table 19: Study 3, Part 1: Intervention and Control Group Students Answers to First Six Scenarios

		Graph 1 	Graph 2 	Graph 3 	Graph 4 	Graph 5 	Graph 6 	Graph 7 
Scenario 1 There are a dozen eggs in each package. As the number of packages increases, the total number of eggs increases at a constant rate	intervention		88.9%		11.1%			
	control	5.6%	83.3%		11.1%			
Scenario 2 As the temperature of the water increases, the amount of oxygen in the water decreases slower and slower	intervention		2.8%	11.1%			75%	11.1%
	control			11.1%			66.7%	22.2%
Scenario 3 As more customers entered the restaurant, the number of available chairs decreased at a constant rate	intervention				2.8%	83.3%	5.6%	.3%
	control			5.6%		72.2%	5.6%	16.6%
Scenario 4 The number of lilipads on the pond increases faster and faster each day.	intervention		11.1%	5.6%	80.5%		2.8%	
	control		22.2%		77.8%			
Scenario 5 The cost to mail a letter in the US is 49 cents, no matter how far it travels	intervention	100%						
	control	94.4%	5.6%					
Scenario 6 As the number of clouds increased, the number of people swimming decreased faster and faster	intervention			2.8%	5.6%	8.3%	5.6%	77.7%
	control			5.6%		11.1%	22.2%	61.1%

Among the students who did not choose the targeted answer each time (see Table 20) a larger proportion of control students (33.3%, in comparison to 11.1% in the intervention group) consistently chose graphs where the direction matched the description. Equal proportion of students in each group (22%) chose graphs with inconsistent attention to direction. Similar to Study 2, a group of intervention students chose answers with rate correct each time, but directionality incorrect, however unlike Study 2, the errors in directionality were not consistent through the entire assessment. There is a third group of students (16.7% of intervention and 22.2% of control students) whose choices did not have a consistent pattern of error.

Table 20: Study 3, Part 1: Response Category of Comparison of Intervention and Control Groups

	Intervention	Control
Chose targeted answer	24 (66.7%)	8 (44.4%)
Direction correct; rate incorrect	4 (11.1%)	6 (33.3%)
Rate accurate; direction incorrect	2 (5.6%)	0
Inconsistencies with respect to both rate and direction	6 (16.7%)	4 (22.2%)

Looking more closely at the answers for students without the target answer each time, there were ten students, four students from the intervention group (11.1%) and six from the control group (33.3%) who consistently chose a graph with the correct directionality each time. For nine of these ten students, their analysis of rate was incorrect on two or more prompts. For one of these ten students, the analysis of rate was incorrect on only one of the prompts; this student was in the control group.

In the intervention group, two students (5.6%) showed errors related to direction only. These two students chose graphs where rate was correct each time but direction was not. One of these students had one error and the other had two errors. It is worth noting that this, again, only happened in the intervention group. However, unlike the results of Study 2, the directionality error was not consistent throughout the entire assessment.

The remaining ten students, six from the intervention group (16.7%) and four from the control group (22.2%) chose two or more incorrect graphs and their choices did not seem to have a pattern as they represented both incorrect direction and rate, or one or the other incorrect.

Part 2: Writing own verbal description.

In the second part of the assessment, students were asked to provide a verbal description for two graphs. In analyzing this data, it became more obvious that the change in scenario each time, each one relatively unfamiliar to the students, impacted their ability to interpret the data effectively. In Studies 1 and 2 the language of the verbal prompts was consistent and it seemed that students modeled their choice of words and sentence structure from the first part of the assessment. In this study, the language in the first part of the assessment varied with each scenario and the language that students produced in this part of the assessment was overall less complete than in previous assessments. Very few students made reference to rate of change in their verbal descriptions.

Each response was classified into levels 0, 1, 2, or 3 (see Table 20). Examples of each type of response level appear in table 21. Level 3 responses accurately and completely described the directionality and the rate of change. Level 2 responses have the correct directionality of the data but have missing or incorrect description of the rate of change. Level 1 responses for question 7 had a correct directionality of the relationship between the variables but mixed up the independent and dependent relationship. The most common error was to insert something extra about the variable days. Level 1 responses for question 8 had an incorrect directionality, but the assignment of dependent and independent variable was correct. A Level 0 response had incorrect or missing description of both direction and rate. This category also includes responses that did not respond to the prompt on the axis in any meaningful way.

Table 21: Study 3, Part 2: Sample Leveled Responses to Question 7

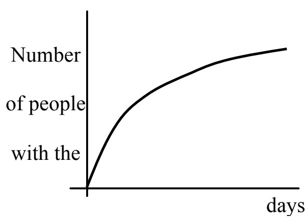
Question 7  	Level 3: At first people rapidly got the flu then less and less got it each day
	Level 2: The number of people with the flu increases every day
	Level 1: As more people got the flu the more days they were out
	Level 0: As the days increase, the number of people with the flu decreased

Table 22: Study 3, Part 2: Number of Responses at each Level for Intervention and Control Groups for Question 7

	Level 3	Level 2	Level 1	Level 0
Intervention	6 (16.7%)	15 (41.7%)	4 (11.1%)	11 (30.5%)
Control	5 (27.8%)	5 (27.7%)	3 (16.7%)	5 (27.8%)

Over half of each group of students was able to achieve a level 2 or 3 response to the prompt. Fewer students than in the previous studies adequately described both directionality and rate of change. Additionally students seemed to have more difficulty determining the way to phrase the sentence to represent the independent and dependent variables relationship.

Table 23: Study 3, Part 2: Sample Leveled Responses to Question 8

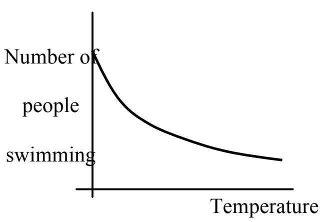
<table>
<tr><td rowspan="4">Question 8

</td><td>Level 3:
As the temp of the ocean rose the amount of people leaving slowed down</td></tr>
<tr><td>Level 2:
As the ocean heats up less people chose to swim</td></tr>
<tr><td>Level 1:
As the ocean got colder people rapidly stopped swimming</td></tr>
<tr><td>Level 0:
As the number of people decreases, the ocean temp evens out</td></tr>
</table>

Table 24: Study 3, Part 2: Number of Responses at each Level for Intervention and Control Groups for Question 8

	Level 3	Level 2	Level 1	Level 0
Intervention	2 (5.6%)	6 (16.7%)	20 (55.5%)	8 (22.2%)
Control	1 (5.6%)	2 (11.1%)	7 (38.9%)	8 (44.4%)

Similar to the results from question 7, the students' showed difficulties in accurately using language to describe both directionality and rate of the graph. Less than half of the students in each group were able to do this. Part of the difficulty may have been related to the fact that the notion that as the ocean water was warmer less people went swimming was contrary to their developed schema of how this relationship works.

Each individual student's responses were next added giving each student a score between 0 and 6. The average for each group is listed in Table 24.

Table 25: Study 3, Part 2: Average Scores for Intervention and Control Groups

Group	Average Score (maximum 6)
Intervention	2.5
Control	2.33

Even though the intervention students performed slightly better than the control students, the difference between the two groups' test scores was not statistically significant (Mann-Whitney $U = 324$, $z = .01$, $p>.4$). Two students in this study (one from each of the control and intervention groups) scored 3 points on both questions. Unlike the two previous studies, where the questions from the first part built a frame of reference for how to respond to these questions, this assessment did not.

Part 3: Sketching a graph given a written description.

In the third part of the assessment, students were asked to sketch a graph to coincide with the verbal description. They were given a written description and labeled axes. Each of their drawn responses was given a score of 0, 1, or 2. A score of 2 was given to sketches that correctly showed directionality and rate and a score of 1 to sketches with incorrect rate but correct direction. All other responses were scored as 0.

Table 26: Study 3, Part 3: Number of Responses at each Level for question 9 and 10

		Level 2	Level 1	Level 0
Question 9	Intervention	24 (66.7%)	12 (33.3%)	0
	Control	8 (44.4%)	9 (50%)	1 (5.6%)
Question 10	Intervention	29 (80.6%)	4 (11.1%)	3 (8.3%)
	Control	11 (61.1%)	4 (22.2%)	3 (16.7%)

Intervention students performed slightly better than the control students, but the difference between the two groups' test scores was not statistically significant (Mann-Whitney U = 324, z = .01, p > 0.4).

Table 27: Study 3, Part 3: Average Scores for Intervention and Control Groups

Group	Average Score (maximum 4)
Intervention	3.38
Control	2.83

Similar to data for questions 1 to 6, the intervention group slightly outperformed the control group. Twenty-eight students scored at level 2 for both questions, 21 from the intervention group (58.3%) and eight from the control group (44.4%). There were no students in either group that scored a 0 on both questions. There were only three students from the intervention group (8.3%) and four from the intervention group (22.2%) that scored a 0 on one of the questions.

Discussion

Results regarding the long term benefits of adding a relationship unit to the Algebra 1 curriculum are somewhat inconclusive from this assessment, since the students from the intervention group only slightly outperformed their control group peers in each part of the assessment. It is worth noting, however, that both groups performed at high levels in the first part of the assessment.

The first possible variable contributing to the somewhat similar performance of the two groups relates to how the teaching of the new Algebra 1 curriculum has evolved in the school over the last few years. From the time of Study1 to the time of Study 3, the school had a large turnover of math teachers and 13 of the intervention students (36.1%) had been taught Algebra 1 by a teacher in her first year who was new to the school and new to the curriculum.

A second factor to be considered in analyzing the data regarding the first part of the assessment is the change in school culture. The Algebra 1 program, now in its fourth year of implementation, has begun to have an impact on the way the entire department approached mathematics. Even though the Algebra 2 classes were taught by the same teachers who had taught them during Studies 1 and 2, the new curriculum ideas about mathematics and about pairing the graphical and alpha-numeric representations led to a paradigm shift in the entire department. Perhaps the similarity between the intervention and control group is representative of this paradigm shift of the entire program. The classes taught at this school are individualized and students have the opportunity to participate in the creation and discussion of mathematics in their classroom. As topics in Algebra 2 are discussed, students from the intervention group may bring their understanding of algebra from the intervention with them to the classroom and this may affect the entire group as they take Algebra 2.

A third factor to consider in analyzing this data is the interaction between the assessment and the student. This assessment was administered language-based learning disabled students exclusively. Their LD diagnosis may include difficulty with language and sentence structure. In studies 1 and 2 the assessment involved very routine language based on time-distance using a consistent model of train movement. In the assessment for this study, the language structure was irregular, as it changed with each scenario, making it harder to model in the second part of the assessment. Additionally, the shifting scenarios required the student each time to create a new image for the situation. Both of these aspects allowed the language disability to impact the student's ability to answer the assessment.

Despite the somewhat similar results between intervention and control groups, it is my belief that this study represents a shifting trend in the culture of this school. The results of the students in each part of the assessment were more similar to the results of the students in the intervention group, after the intervention, in Studies 1 and 2. This may be a result of the design of the assessment or it may be the result of a paradigm shift at the school. Through professional development opportunities within the mathematics program, teachers at this school were changing their methods of approaching algebra. Algebra 1 teachers from Study 1 had changed their teaching methods and encouraged conversation, as the department head (author) presented opportunities for teachers to begin to think about mathematics and algebra through a multi-representational lens. Ideally this data would have been taken during the first year of implementation of the program for less muddled results. However the results from this Study are encouraging at documenting a shifting culture of an entire program.

Chapter 8: Conclusions

This dissertation is a longitudinal study of the impact of an algebra program that encourages connections between the graphical, alpha-numeric and descriptive representations of algebra. It is the culmination of many years of teaching experience, the work of a group of motivated teachers, of students in need of a different approach and of access to the research and ideas on a functional approach to algebra. The dissertation documents a small aspect of the change and effect of this program among LD students' understanding. The school where the study was conducted changed its approach and teaching style as it relates to algebra as a result of this work.

An essential aspect of this dissertation is the school curricular change. It changed from a traditional study of Algebra 1, to a study of Algebra 1 that aimed at deep connections between the graphical, alpha-numeric and descriptive representations of algebra. Curriculum is only as good as the teachers who work with it. The four teachers in the initial study replaced the traditional view of algebra teaching focused on alpha-numeric manipulations and began to think about algebra as a multi-representational subject. This shift in thinking about mathematics and algebra was also shared with the other members of the department through author led discussions and Algebra 1 teacher led discussions, as others in the department were intrigued by the changes occurring in the Algebra 1 classrooms. The three studies conducted over the course of four years documents changes in student's representational thinking and suggest that there was also a whole department change on how teachers approached algebra in all algebra classrooms.

Study 1 documents the new algebra curriculum's initial impact on the students. The results of the post-instruction assessment were extremely encouraging. Students were able to make better connections between the verbal description of events and graphs of a variety of functions and relationships. Teachers were excited about what they were teaching; students were engaging in more discussion in their classrooms and the results of the post-instruction assessment documents the substantial growth in student understanding. The inclusion of a control group of Algebra 2 students ruled out the possibility that these skills would develop over time on their own. As a result of the change in instruction, Algebra 1 students were able to:

- accurately recognize directionality
- accurately recognize varying vs. constant rate of change
- more consistently use language to describe directionality
- more consistently use language describe varying vs. constant rate of change

It was further shown that the majority of students at the Algebra 2 level had skills similar to the Algebra 1 students *before* instruction. These skills and abilities do not seem to develop among LD students in a traditional algebra program.

Study 2 documents the program's long-term impact on students' ability to interpret relationship graphs. After the initial exposure to the new Algebra 1 curriculum, students were given the same test two years later, as they were taking an Algebra 2 course and compared to students in the same classes who had not received Algebra 1 instruction in this school. Over 60% of the intervention group maintained their ability to correctly interpret relationship graphs. This compares to 28% of the students in the control group who were able to do the same. An even larger difference between the groups appears when we look at the number of students whose errors seemed to have no distinguishable pattern. Six percent of the intervention group fit this category compared to over half (56%) of the control group. This means that for more than half of the control group students, errors throughout the assessment had no distinguishable pattern and were most similar to those from the original pre-intervention group from Study 1.

The intervention group scores in Study 2, similar to those collected two years earlier, immediately after the intervention, indicates that an initial instruction on relationship and function graphs at the start of the Algebra 1 course stayed with the students with no maintenance or further deliberate instruction over the course of two years. This is a dramatic achievement for a group of LD students who usually have difficulties with memory and retention.

Study 3 looked at the students understanding of graphs of relationships between variables other than distance and time. The data gathered in this case were less conclusive than those from the previous two studies: the results of the intervention groups were higher than those of the control group but the difference was not statistically significant. Table 28 shows that the scores obtained by the control students in Study 3 are more closely aligned to the intervention group from the previous two studies. This may be related to the fact that the mathematics department was in its fourth year of this new Algebra 1 curriculum and its effects were perhaps being extended to other courses. As the approach and discussions of the entire department shifted to include a more multi-representational view of algebra, it is reasonable to expect that some of the effects of these discussions led to changes among the Algebra 2 teachers' views and ways of teaching algebra.

Table 28: Comparing Part 1 of all Three Studies: Average Scores for first part of testing for each of the three studies

Group	Average Score (maximum 12)
STUDY 1 – distance-time graphs	
Algebra 1 Pre-test (intervention)	5.5
Algebra 1 Post-test (intervention)	11.1
Algebra 2 Test (control)	6.8
STUDY 2 – distance-time graphs	
Algebra 2 Intervention	10.42
Algebra 2 Control	7.76
STUDY 3 – relathionship graphs	
Algebra 2 Intervention	10.94
Algebra 2 Control	10.32

The questions guiding the set of three studies were:

a. Can high school Algebra 1 LD students learn to interpret function graphs considering rate and directionality?

b. How do they compare to control LD students at a higher grade and age level immediately after the intervention?

c. Did they retain their ability to interpret function graphs two years after they participated in the intervention?

d. How do they compare to control students at the same grade and age level two years after the intervention?

e. How do they interpret and produce graphs relating different kinds of variables two years after they experienced the new Algebra 1 curriculum?

Answers to the first question are important to determine if the intervention curriculum is age-appropriate. Many years ago, discussions relating to interpreting the subtleties of relationship graphs were left for the start of Calculus. At this school, and around the nation, only the top mathematics students would engage in these discussions in high school classrooms. What the three studies' results show is that the conversation about relationships and functions is perfectly suited as an Algebra 1 foundation topic. The students are capable of managing the discussions and of learning the related skills. Moreover, what they have learned in these classes stayed with them for many years after the initial instruction.

The next three questions led to comparisons between the intervention students and their control group peers, immediately after the new Algebra 1 course and years later. Studies 1 and 2 show that the intervention group students are better able to interpret distance-time graphs than their control group peers both immediately after the intervention and two years later. The intervention groups show more consistency in their ability to interpret graphs as well as in the errors that they make. For instance in Study 2, there was a subset group of students who consistently interpreted direction as opposite to that in the description. This indicates that they understood that direction was important, but had it reversed. These students are in a better position than students showing answers that seem to have no consistent pattern of error, since it is easier to redirect and fix errors related to only one aspect of the graph.

The final question looks at graphs on the relationship of a variety of variables. While the students in Study 3 had more difficulty giving verbal descriptions of the graphs they were looking at, both intervention and control students were able to match graphs to descriptions.

These studies focused on a population of LD learners. LD learners are more susceptible to the nuance of instruction as their limits on working memory, grapho-motor difficulties, misinterpretations of signs and difficulty with rote arithmetic facts make learning algebra through traditional methods difficult. The unique opportunity to work with this group of students is relevant to all students who find difficulty gaining access to algebra through the traditional sequence of the algebra curriculum. The LD population represents about 5% of the public school community but the results of this study support the view that an algebra curriculum rich with multiple-linked representations may be beneficial to all students.

Beginning Algebra 1 with instruction on interpreting the graphical representation of functions, instead of focusing on the alpha-numeric representation of algebra, allows for learning among LD students that remains over the years. One must recognize that this approach to Algebra 1 may delay certain areas of study for later in the year. However, as the courses were implemented, nothing was compromised and students were given access to all of the same skills. What they gained in multi-representational thinking and relationship interpretation stayed with them throughout the following two years. What remains to be examined by future research is how the early experiences with a functional approach to algebra may help students understand other aspects of the algebra curriculum.

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Appendix A - Intervention Curriculum

Algebra is the study of relationships

Unit 1: Relationships

Goal:

- read a problem and pull out the factors in the relationship
- get comfortable with sketching graphs
- lay groundwork for understanding the relationship between slope and rate
- recognize there are limited ways in which relationships can increase/decrease or stay constant

Lesson Plans:

Explicitly stating relationships

- what things are being related? implied relationships and explicit relationships
- describing the relationship

Sketching a graph

- paying attention to what the graph means
- labeling the axes
- working with which variable is dependent on the other

Interpreting a sketched graph

- Describing the relationship between a sketched graph and the problem
- making up stories to describe the graph

Ranger

- relationship between rate and shape of the graph.
- what are the options for graph directions (increase, decrease, constant)

- with one walking plan, how many different ways can you make the graph increase? decrease? stay constant?

Sample Schedule

In class	Day 1: Pre-test	Day 2: Relationships Individual cards Classwork 1	Day 3: Matching graphs to situations Sketching a graph of relationships Classwork 2	Day 4: Wipe board - Sketching and drawing graphs of distance/time relationships	Day 5: Ranger Walk the line
HW	Skill maintenance	Homework 1, start in class	Homework 2, start in class	Homework 3, matching graphs to problem and skill maintenance	Skill maintenance

	Day 6: Ranger – Walk the line	Day 7 : Different ways to increase; running 200 m race (visual math website) describing different increasing relationships	Day 8: Wipe board – story problems Getting at the root of different ways on increasing, decreasing…	Day 9: Mid Test and class interview	
	Skill maintenance	Homework 4, walk the line worksheet	Homework 5, Review of skills		

Running a Marathon	The number of raisins in a box	The price of a movie ticket
Minutes it takes to get to class	Amount of water in a bucket	Money made per hour worked
Cost of gas	Height of water in a bucket	Cost of fence
Cost of mulch needed	Value of home	*Relationship cards*: In pairs discuss what two variables should be considered when working with this problem, make up one of your own as well

Algebra x

x

The Study of Relationships – ***Classwork 1***

Relationship: ____

Variables to consider:

Relationship: ____

Variables to consider:

Algebra x

x

The Study of Relationships – ***Classwork 2***

Relationship:____

variables: ________________

____

rate: ________________ / ________

Relationship:____

variables: ________________

____

rate: ________________ / ________

Algebra x

x

The Study of Relationships – ***Classwork***

Examine each pair of variables. Write whether the variables show a positive relationship, negative relationship, or no relationship.

1. Number of overtime hours worked and amount of income
2. Weight of a vehicle and number of miles it gets to a gallon of gasoline
3. Weight of a vehicle and number of gallons of gasoline its tank holds
4. Number of miles within the United States that a first class letter is sent with the U.S. Postal Service and cost of postage
5. Heights of students and number of miles they live from school
6. Amount of money spent on a car and color of the car
7. Students' test scores on reading and math tests
8. Shoe size and heights of teenagers
9. Social studies test results and number of hours of television watched the night before the test
10. Number of minutes played and number of points scored in a basketball game

Algebra x

x

The Study of Relationships – ***Homework 1***

1. An Olympic runner runs the 200 meter dash.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

2. Your hair grows longer each day.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

3. There are 21 m&m's in each little bag.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

4. The amount of shampoo in your shampoo bottle.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

5. A waitress earns about $15 per table she waits on.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

6. 4 people can be seated at each table.
   variables: ____________ ____________

   relationship: As ________________ increases ________________ ______________.

   variable *variable* *increases/decreases*

7. The value of a car depreciates over time.
   variables: ______________ ______________

   relationship: As __________________ increases __________________ _______________.

   variable *variable* *increases/decreases*

8. When Mrs. Sauriol was in high school, movie tickets cost $3.75 each!
   variables: ______________ ______________

   relationship: As __________________ increases __________________ _______________.

   variable *variable* *increases/decreases*

Make up 2 of your own relationship statements.

9. ____.
   variables: ______________ ______________

   relationship: As __________________ increases __________________ _______________.

   variable *variable* *increases/decreases*

---

10. ____.

    variables: ______________ ______________

    relationship: As __________________ increases __________________ _______________.

    variable *variable* *increases/decreases*

Algebra x
x

The Study of Relationships – ***Homework 2***

1. An Olympic runner runs the 200 meter dash.

variables: ____________________

____

rate: ____________________ / ____________________

2. Your hair grows longer each day.

variables: ____________________

____

rate: ____________________ / ____________________

3. There are 21 m&m's in each little bag.

variables: ____________________

____

rate: ____________________ / ____________________

4. The amount of shampoo in your shampoo bottle.

variables: ____________________

____

rate: ____________________ / ____________________

A waitress earns about $15 per table she waits on.

variables: ____________________

____

rate: ____________________ / ____________________

5. 4 people can be seated at each table.

variables: ____________________

____

rate: ____________________ / ____________________

6. The value of a car depreciates over time.

variables: ____________________

____

rate: ____________________ / ____________________

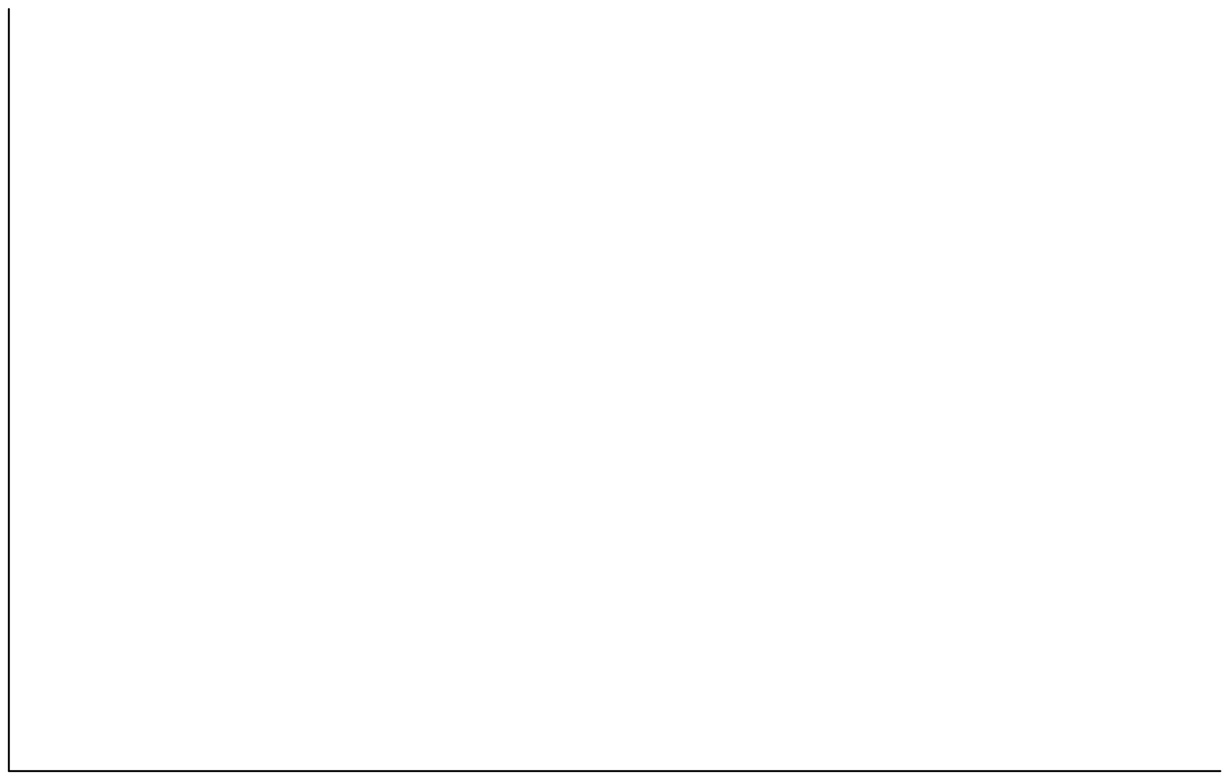
7. When Mrs. Sauriol was in high school, movie tickets cost $3.75 each!

variables: ____________________

____

rate: ____________________ / ____________________

Story Graphs



Algebra x

x

The Study of Relationships *- Homework 3*

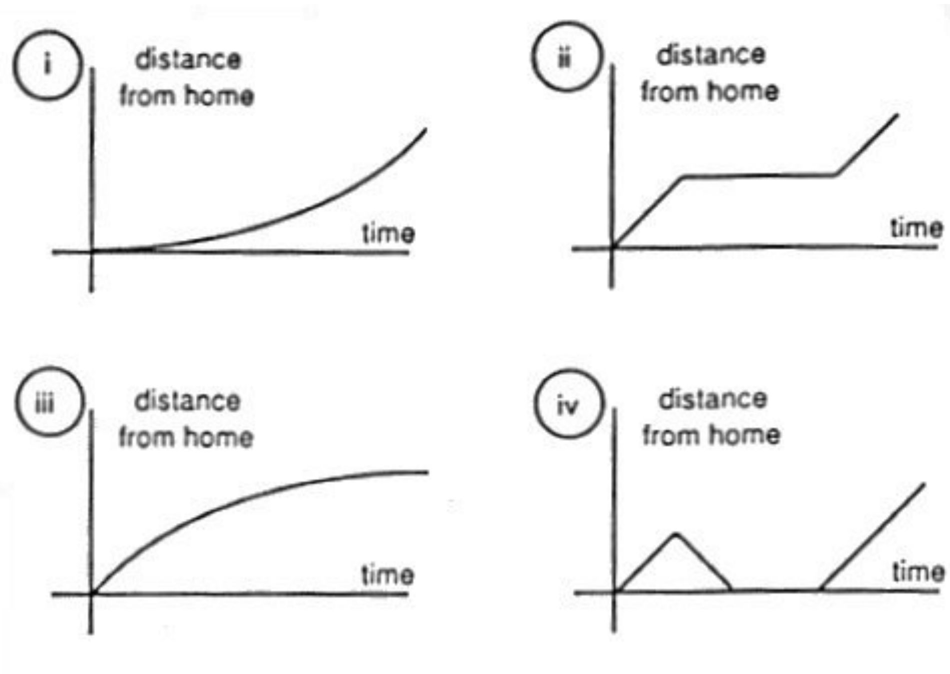


Figure 1. Which story goes with which graph?

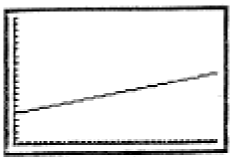
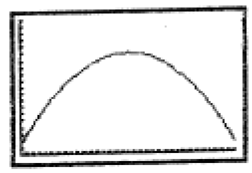
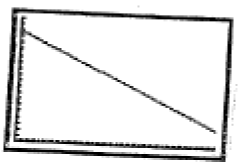
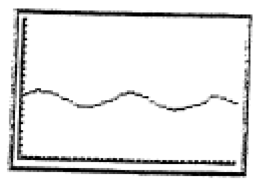
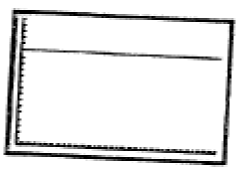
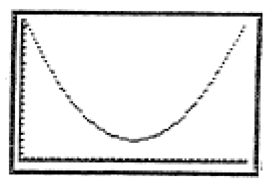
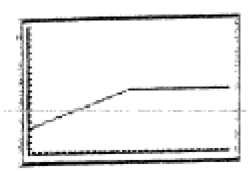
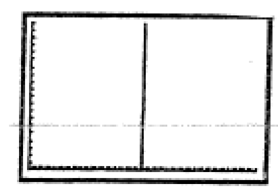
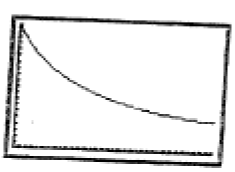
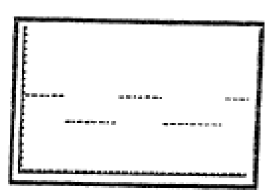
Match the stories with three of the graphs above and write a story for the remaining graph.

(a) I had just left home when I realized I had forgotten my books, so I went back to pick them up.

(b) Things went fine until I had a flat tire.

(c) I started out calmly but went faster and faster as I realized I was going to be late.

Walk the Line

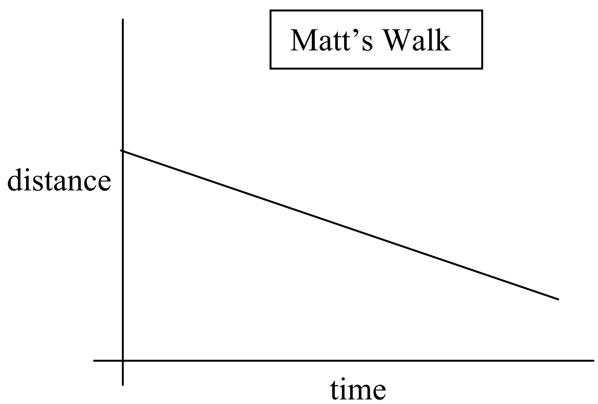
Graph to Walk	Your Walking Strategy	Graph to Walk	Your Walking Strategy
1. 		6. 	
2. 		7. 	
3. 		8. 	
4. 		9. 	
5. 		10. 	

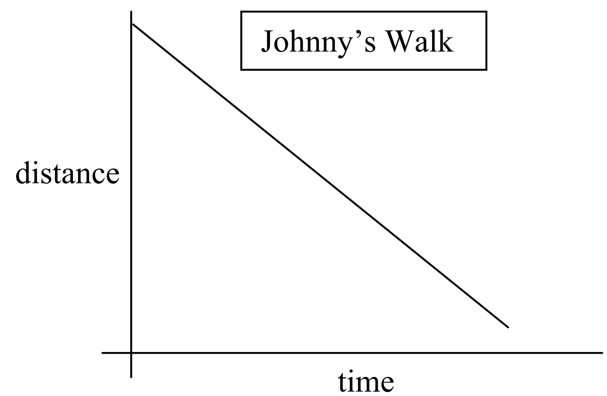
Algebra x

x

The Study of Relationships *- Homework 4*

2 different students produced 2 different ranger graphs (represented below).





1st, describe each of their walking strategies.

______________________ ______________________

______________________ ______________________

______________________ ______________________

2nd, answer these questions.

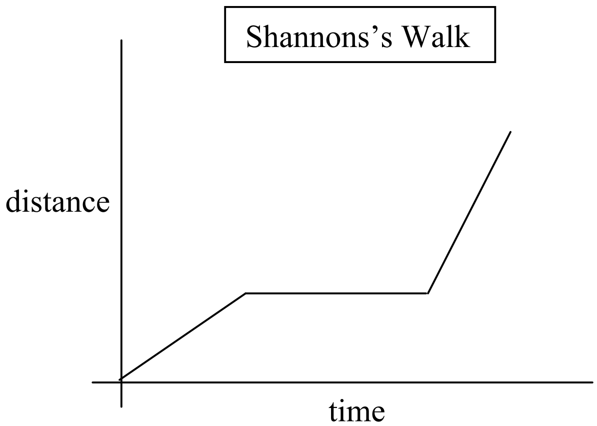
Who was walking more quickly? ______________________

How do you know?

Who started closest to the ranger? ______________________

How do you know?

Another student walked and had this ranger graph.



Describe her walking strategy.

There seem to be three segments to this graph. During the middle segment, what was she doing?

During which segment was she walking the fastest? ___________________________________

How can you tell?

Algebra x

x

The Study of Relationships *- Homework 5*

Mrs. Sauriol went to visit a friend who lives in Providence this weekend. She drove from Landmark to Rhode Island in 3 hours.

Show at least three different distance vs. time graphs of what her trip could have looked like.

Tell the story of her trip as represented by each graph.

Scenario 1:

Scenario 2:

Scenario 3:

Draw a graph for the story that is presented:

Scenario 4:

Mrs. Sauriol left home, she was about 5 miles from home when she realized she had left the directions on her table, she went back and got them. Initially the traffic was slow due to construction and it took a long time for her to get through Boston. When she was about 2/3 of the way there, the traffic cleared and she was able to travel more quickly to meet her friend.

Unit 2: Analyzing and creating functions

Goal:

- Realize that there are different functions (other than linear)
- Explore different functions in both the graphical and algebraic representation
- Use function notation

Lesson Plans:

Function notation

- Evaluating algebraic expressions
- Using function notation
- Understanding what the calculator/computer does to graph a function
- Evaluating a function and plotting points on the Cartesian coordinate system

Categorizing functions

- Based on their shape
- Based on their algebraic representation

Looking at properties of different functions

- Making a good sketch
- Increasing/decreasing… what are the ways they increase and decrease
- If you saw part of a graph, could you tell some properties about that graph?
- What does changing the leading coefficient do? Changing the added constant?
- Graph matching

Sample Schedule

In class	Day 1: Evaluating Algebraic Expressions	Day 2: Creating a table	Day 3: Graphing a function	Day 4: Using function notation	Day 5: Quiz Using function notation
HW					

	Day 6: Function Card Sort	Day 7 : Computer exploration of functions	Day 8: Describing Shapes of Graphs	Day 9: Exploring functions	Day 10: Post-test
HW					

Algebra x

x

Function Notation

f(x) = 2x + 7

Find:

f((5) = f(-2) =

f(1.5) = f(½) =

f(x) = 5x - 4

Find:

f((3) = f(-7) =

f(3.2) = f(¼) =

Algebra x

x

Function Notation

$f(x) = 3x^2 + 2(x - 9)$

Find:

$f(3) =$ $f(-4) =$

$f(2.5) =$ $f(\frac{1}{2}) =$

$g(x) = 5x + 9(2+x) - 17$

Find:

$g(4) =$ $g(-10) =$

$g(100) =$ $g(.5) =$

Given the two functions above, find:

$f(3) + g(4) =$ $f(10) + g(10) =$

Algebra 2 Homework x

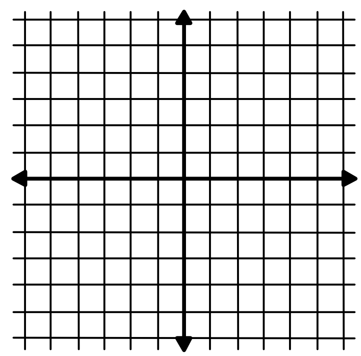
Graphing Functions x

Make a table of values, then graph the following functions (they are not all linear)

---

1. y = 3x + 6

x	y = 3x + 6	(x, y)
-5		
-3		
-1		
0		
1		
3		



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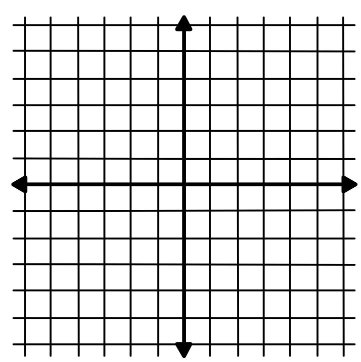
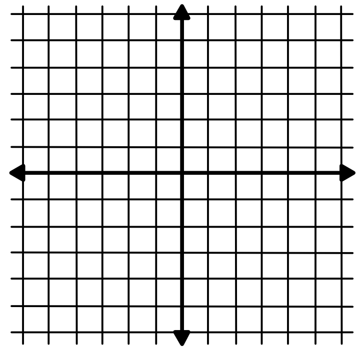
2. y = -x – 8

x		(x, y)

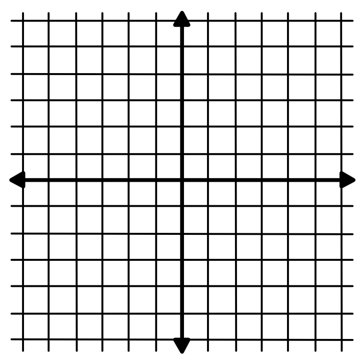


$y = x^2 – 3x$



x		(x, y)

3. $y = 3x^2 – 2$

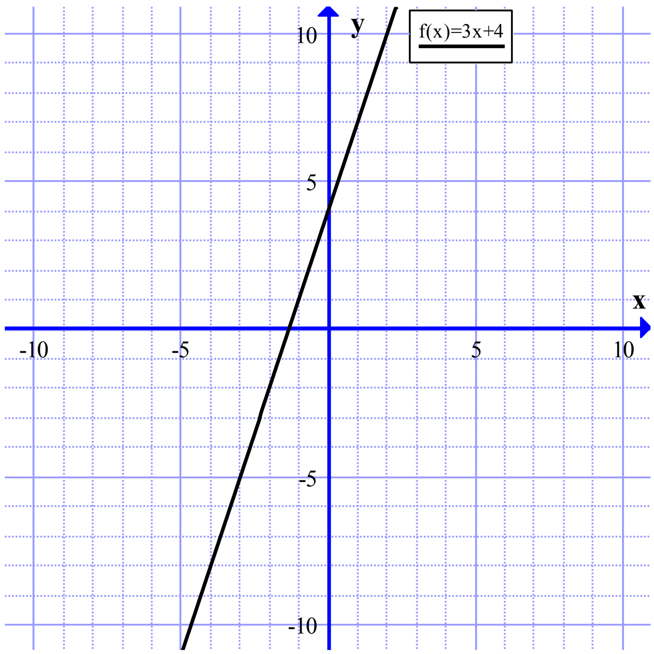


x		(x, y)

Graph Reading x

Read the graph and complete to make true statements.

1.



f(x) = 3x + 4

f(2) =

f(-1) =

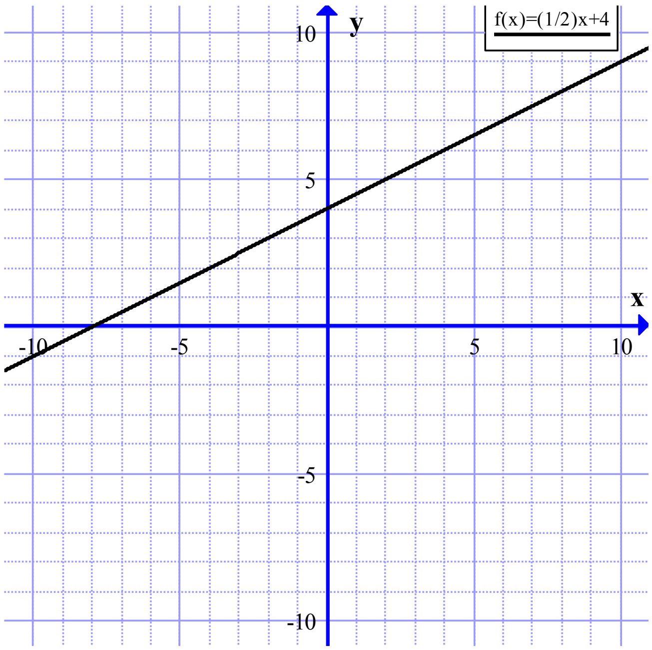
f(-2) =

f(0) =

f(   ) = 10

f(   ) = -5

2.



f(x) = ½ x + 4

f(5) =

f(-4) =

f(10) =

f(0) =

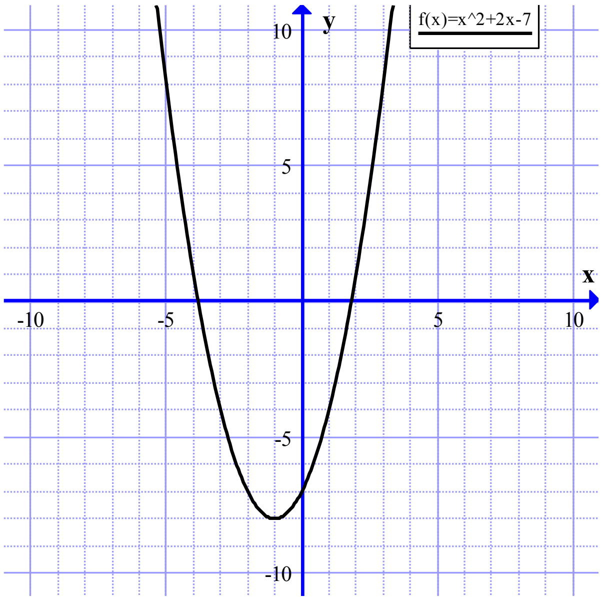
f(   ) = -1

f(   ) = 5

Make 3 observations about problems 1 and 2:

-
-
-

3.



$f(x) = x^2 + 2x - 7$

f(2) =

f(-5) =

f(1.5) =

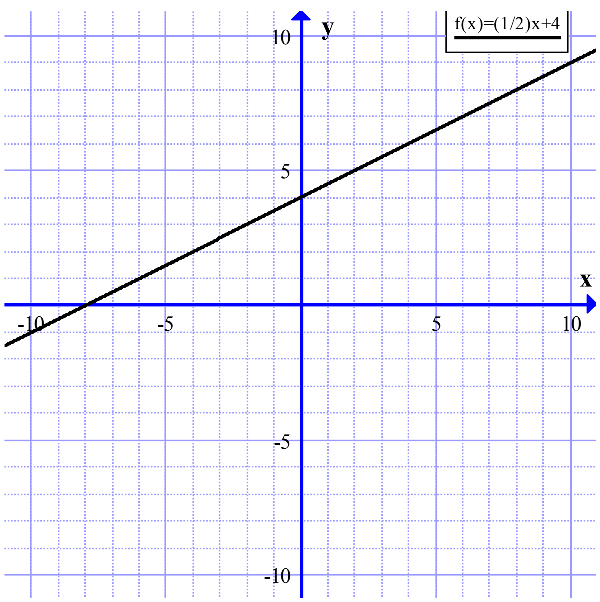
f(0) =

f(   ) = -8

f(   ) = -4

**how many possible answers are there?

4.



f(x) = ½ x + 4

f(5) =

f(-4) =

f(10) =

f(0) =

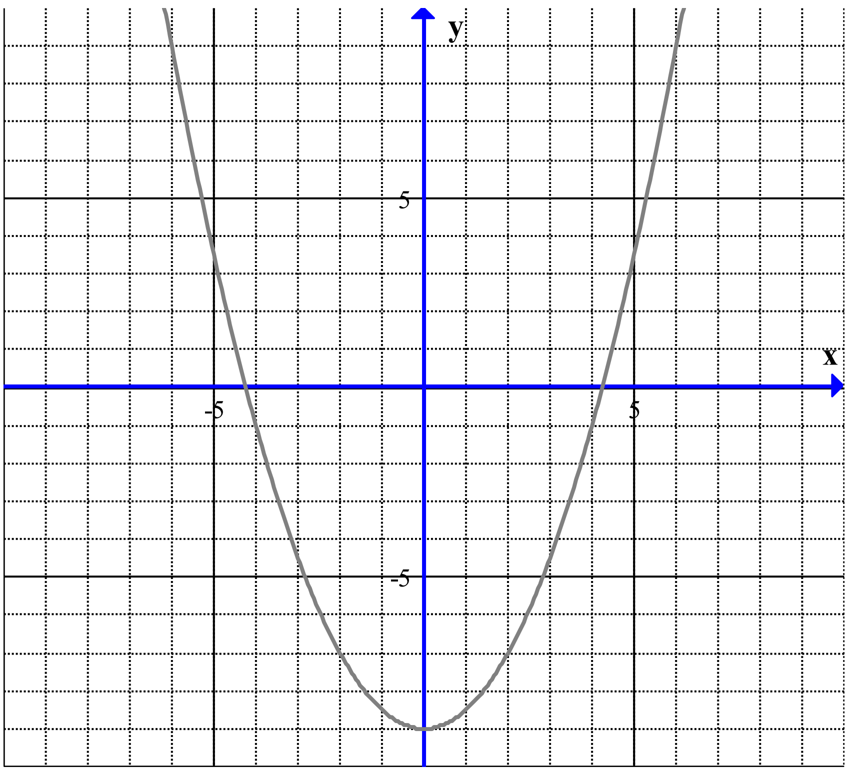
f(   ) = -1

f(   ) = 5

Evaluating Functions

x
x

Classwork



Make 3 observations about this graph:

-
-

Find the missing values.

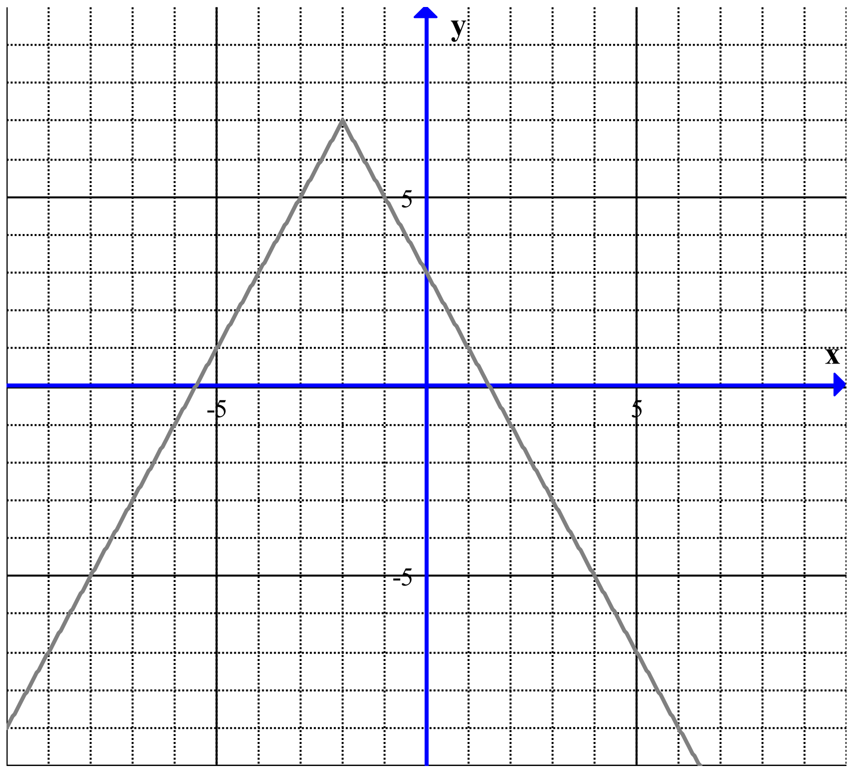
f(0) =	f(2) =
f(4) =	f(6) =
f(-2) =	f(-4) =
f(-6) =	f(1) =
f(5) =	f(-5) =
f( ) = -1	f( ) = -8.5
f( ) = -1	f( ) = -8.5

Evaluating Functions

x

x

Homework



Make 3 observations about this graph:

-
-

Find the missing values.

f(0) =	f(1) =
f(5) =	f(-2) =
f(-6) =	f(3) =
f(-7) =	f(-3) =
f(7) =	f(-4) =
f( ) = 7	f( ) = -5
f( ) = 7	f( ) = -5

$$\boxed{f(x) = |x+2| - 3}$$

$f(3) = |3+2| - 3$

$=$

$=$

$f(-3) = |-3+2| - 3$

$=$

$=$

$f(0) =$

$f(-5) =$

$f(5) =$

$f(-2) =$

Without calculating, look at your sketch and estimate:

$f(-1) =$

$f(-4) =$

$f(1) =$

$f(2) =$

$f(1.5) =$

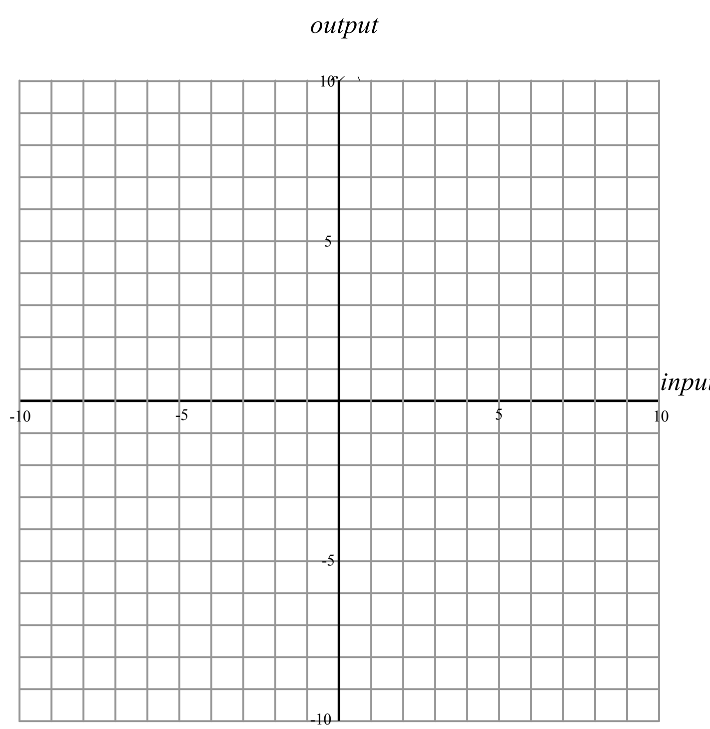
$f(10) =$

$f(-10) =$

$f(\quad) = 1$

$f(\quad) = 1$

input (x)	output f(x)
3	
-3	
0	
-5	
5	
-2	



How many different functions do you remember from yesterday???

→

→

$f(x) =$

→

→

$f(x) =$

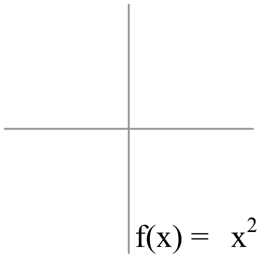
→

→

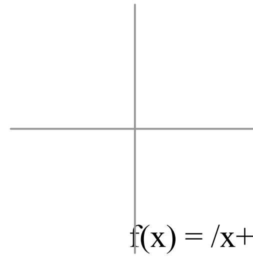
$f(x) =$

Functions

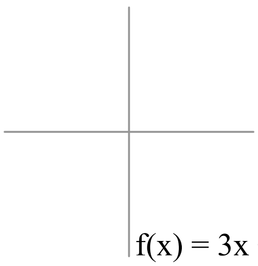
Using the computer, Sketch these functions:



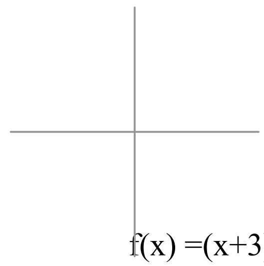
f(x) = $x^2 + 2x - 3$



f(x) = /x+1/-7

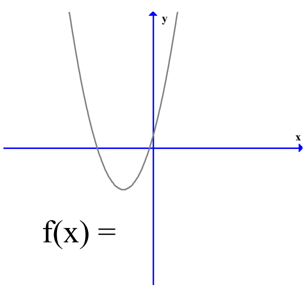


f(x) = 3x + 9

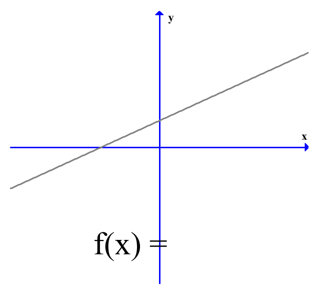


f(x) =$(x+3)^3+5$

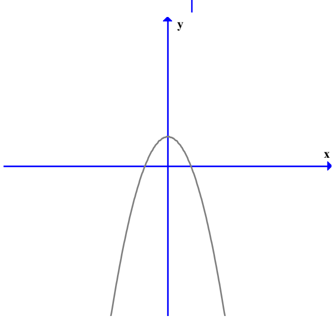
Create a function that looks like this:



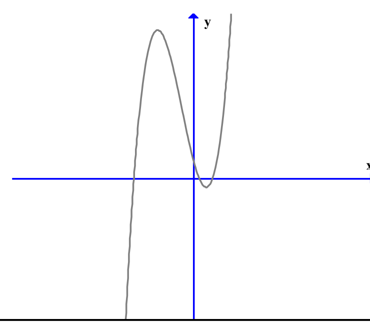
f(x) =



f(x) =



f(x) =



f(x) =

Two things I found interesting were:

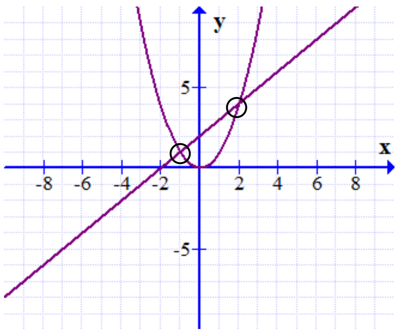
-
-

Algebra x

x

How many intersections?

Example:

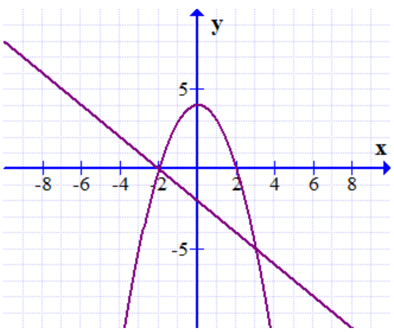


This function has _2_ points of intersection.

They are: x = -1

x = 2

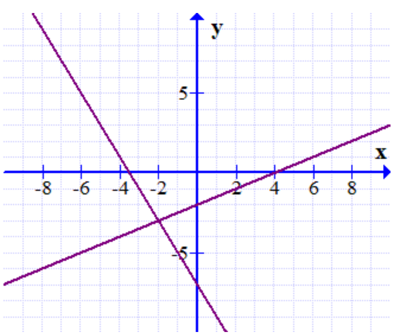
1.



This function has _____ points of intersection.

They are:

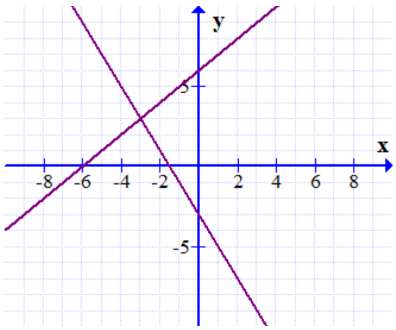
2.



This function has _____ points of intersection.

They are:

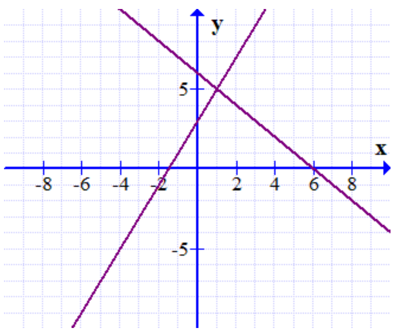
3.



This function has _____ points of intersection.

They are:

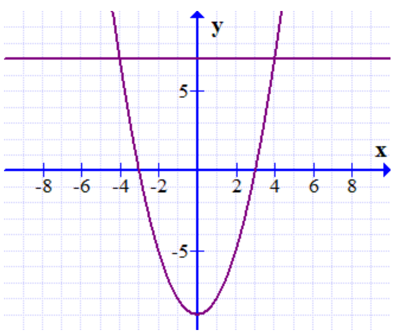
4.



This function has _____ points of intersection.

They are:

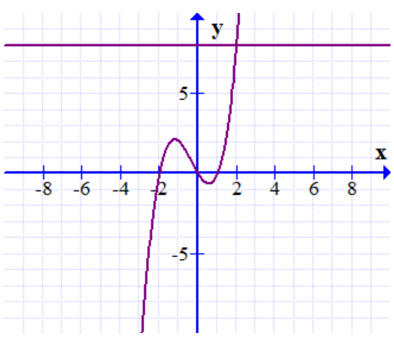
5.



This function has _____ points of intersection.

They are:

6.



This function has _____ points of intersection.

They are:

Appendix B – Assessments

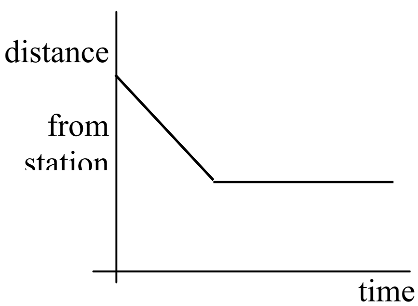
Distance-Time Relationship Test

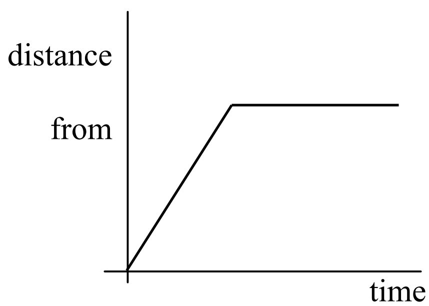
Algebra 1 - Functions

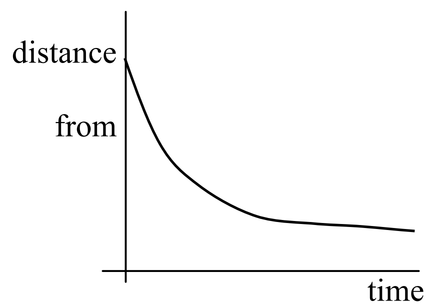
Name:

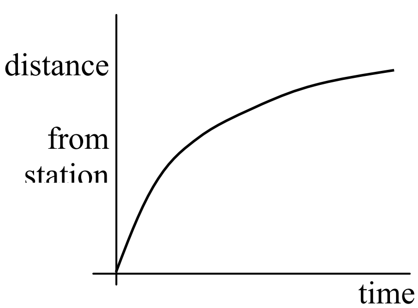
Date:

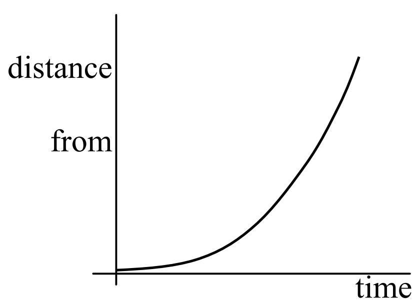
Day:

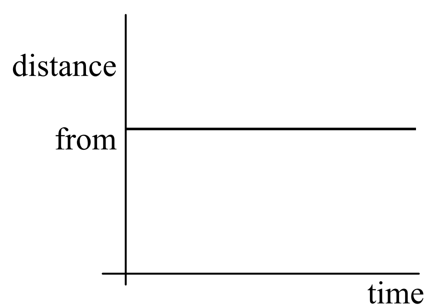


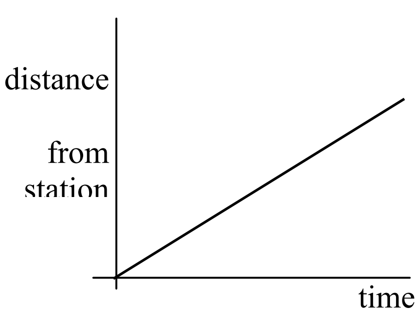


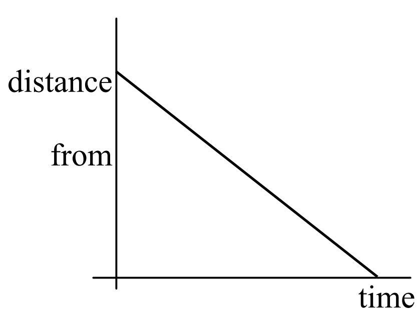












A train's movement along a track is shown in the graphs. Distance, represents distance from the station at a certain time.

Match each graph to a story.

1. The train left the station and moved away from the station at a constant rate.	2. The train was stopped on the tracks
3. The train was moving faster and faster away from the station	4. The train was coming into the station at a constant rate.
5. The train was coming into the station at a constant rate and then stopped.	6. The train left the station and then suddenly stopped.

You should have two graphs left. Create a story about the movement represented in those two graphs.

7. ______________________________

8. ______________________________

Sketch a graph for the stories below.

9. A train left the station at a steady rate. It stopped suddenly for 2 minutes and then resumed at the same steady rate

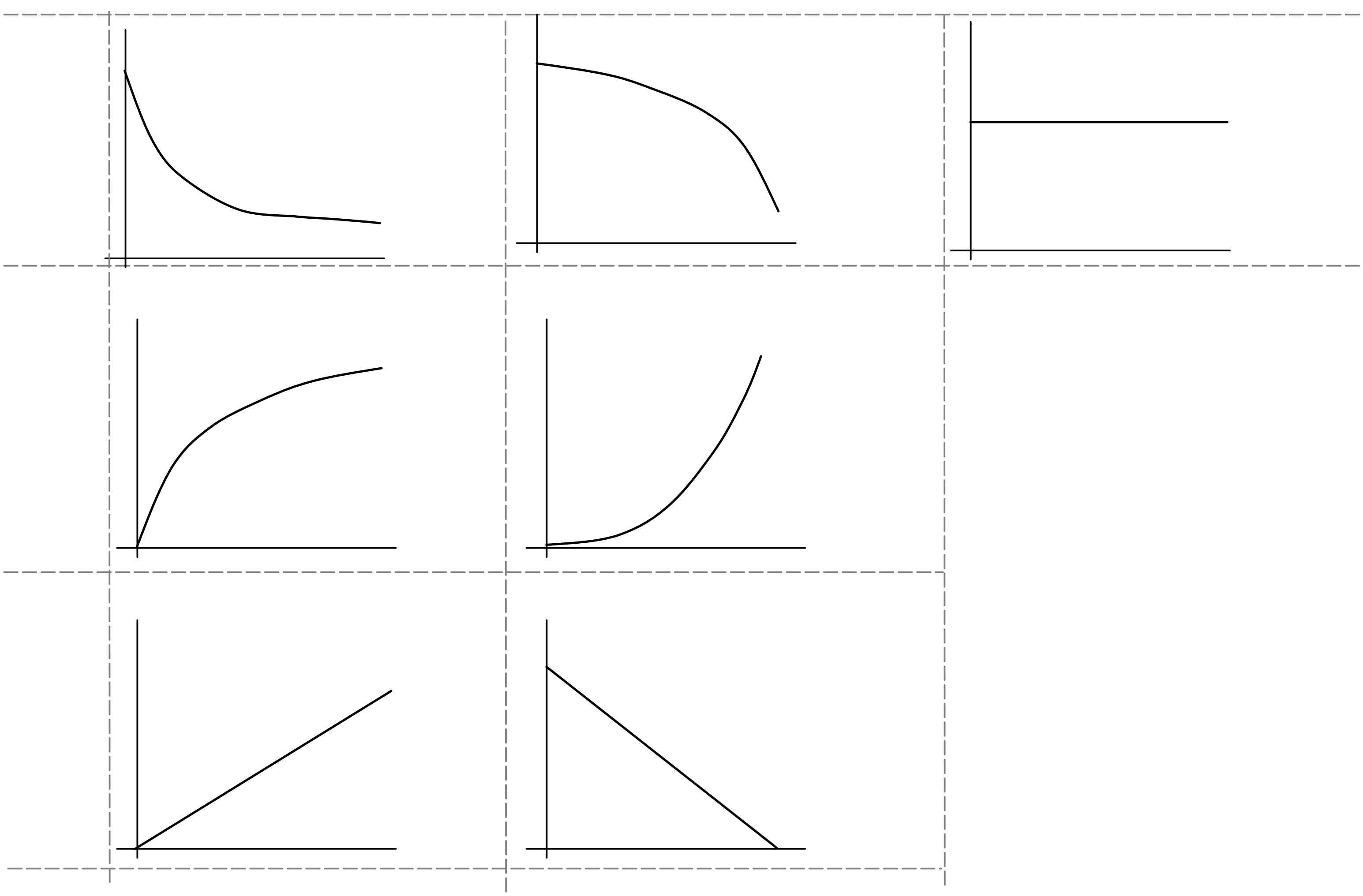
10. A train was about 2 miles from the station when it began to move away from the station at a constant speed.

Relationship Assessment

Relationships

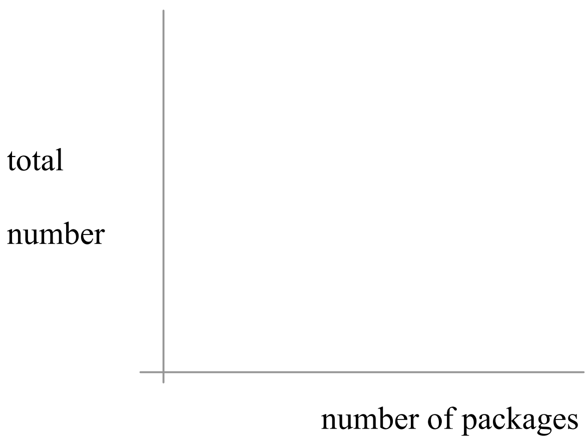
Name: ____

Date: ____

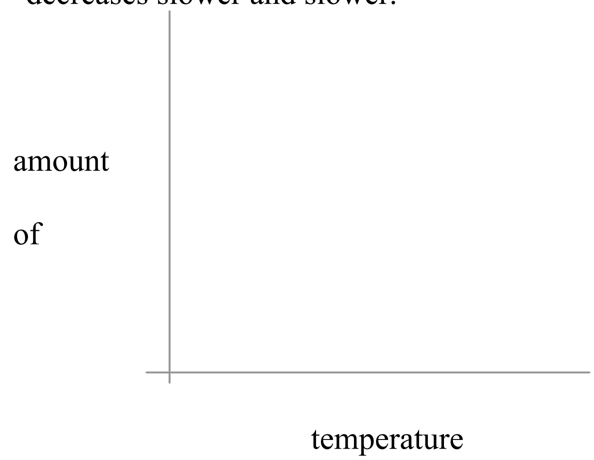


Given each scenario, choose the best graphical representation of the relationship. Pay attention to the labels on the axis. (You will have one graph left over)

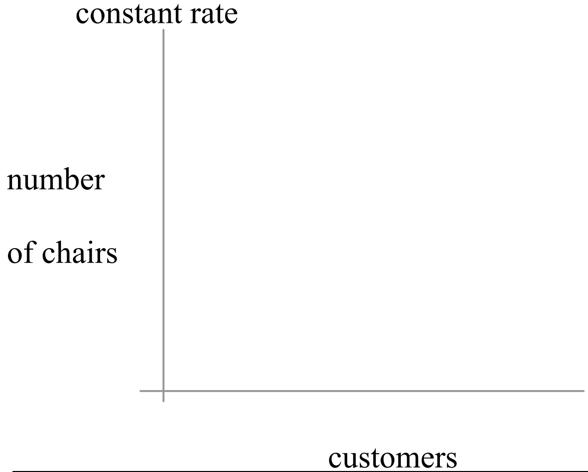
1. There are a dozen eggs in each package. As the number of packages increases, the total number of eggs increases at a constant rate.



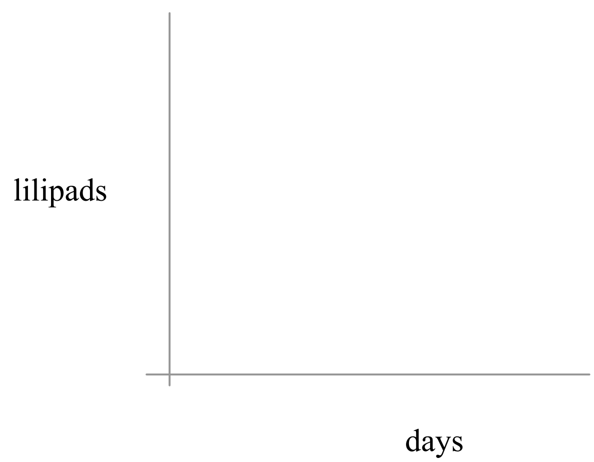
2. As the temperature of the water increases, the amount of oxygen in the water decreases slower and slower.



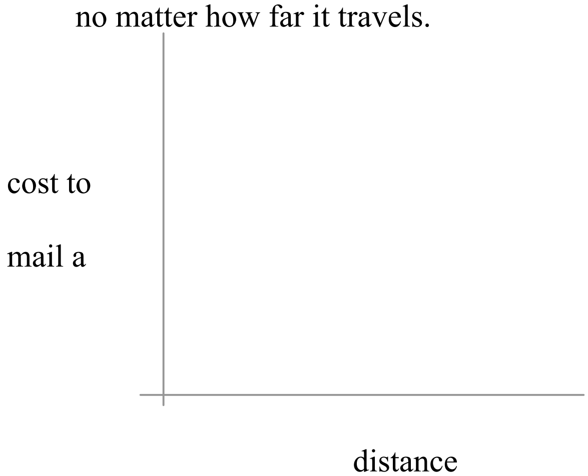
3. As more customers entered the restaurant, the number of available chairs decreased at a constant rate



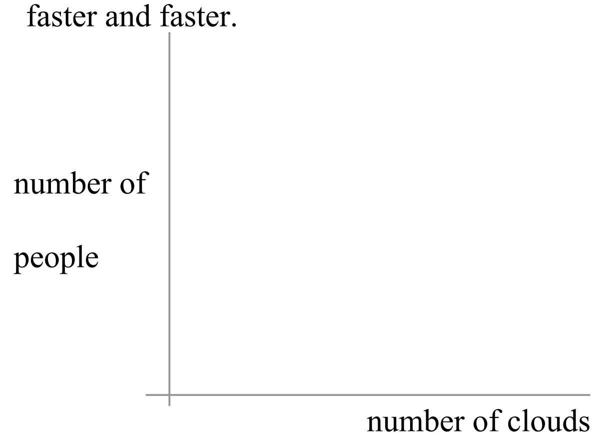
4. The number of lilipads on the pond increases faster and faster each day.



5. The cost to mail a letter in the US is 49 cents, no matter how far it travels.

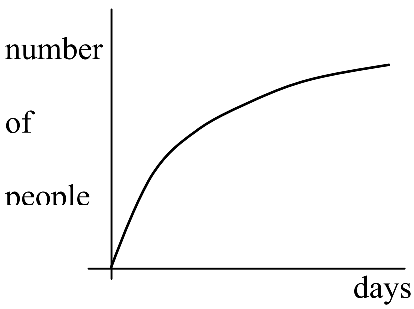


6. As the number of clouds increased, the number of people swimming decreased faster and faster.

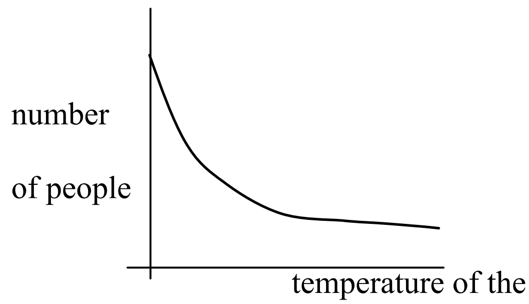


Create a story about these two graphs

7.

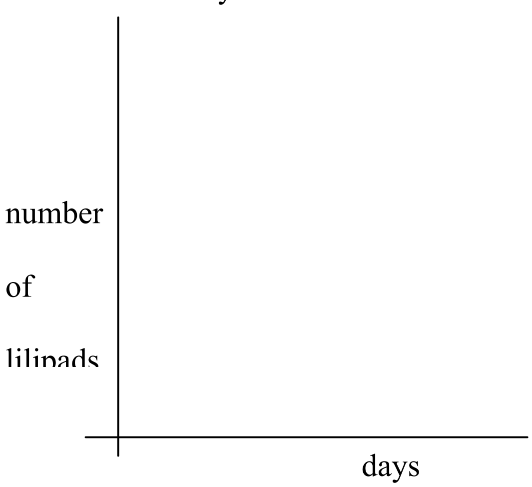


8.

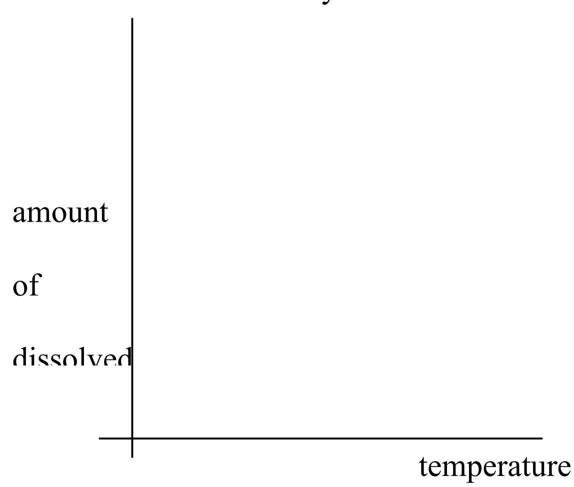


Sketch a graph for the stories below.

9. The number of lilipads on the pond decreased at a constant rate each day and then one day stopped decreasing and stayed the same.



10. As the temperature of the water increased, the amount of salt that could be dissolved into the water increased at a steady rate.



Appendix C – Study 1 Results

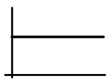
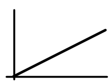
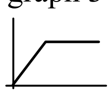
Study 1, Part 1: Individual Scores for Algebra I pre- and post- and Algebra II

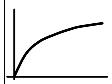
Algebra I	Pre-	Post-
Student 1	2	12
Student 2	8	10
Student 3	5	12
Student 4	10	12
Student 5	5	12
Student 6	4	12
Student 7	4	9
Student 8	3	12
Student 9	2	7
Student 10	5	12
Student 11	12	12
Student 12	3	10
Student 13	1	9
Student 14	5	12
Student 15	1	12
Student 16	12	12
Student 17	1	12
Student 18	3	10
Student 19	1	9
Student 20	10	12

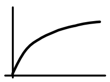
Algebra I	Pre-	Post-
Student 21	12	12
Student 22	8	10
Student 23	3	12
Student 24	12	12
Student 25	12	12
Student 26	5	9
Student 27	5	12
Student 28	4	10
Student 29	2	12
Student 30	8	12
Student 31	3	6
Student 32	3	12
Student 33	1	12
Student 34	2	9
Student 35	10	12
Student 36	6	12
Student 37	10	12
Student 38	5	12
Student 39	10	12
Student 40	4	12

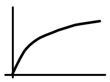
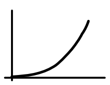
Algebra II	
Student AII 1	12
Student AII 2	12
Student AII 3	8
Student AII 4	12
Student AII 5	12
Student AII 6	1
Student AII 7	8
Student AII 8	4
Student AII 9	4
Student AII 10	7
Student AII 11	4
Student AII 12	1
Student AII 13	7
Student AII 14	10
Student AII 15	7
Student AII 16	4
Student AII 17	3
Student AII 18	4
Student AII 19	7
Student AII 20	10
Student AII 21	8
Student AII 22	6
Student AII 23	8
Student AII 24	4
Student AII 25	7

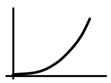
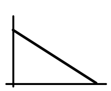
Study 1, Part 2: Written Responses Sorted by Category

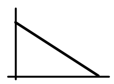
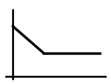
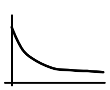
	Algebra I pre-test	Algebra I post test	Algebra II test
graph 1 	*Both descriptors incorrect* – the train was going straight	*Both descriptors correct* – the train never moved – the boat was stranded in the middle of the ocean for many hours not moving at all – the train didn't move	
graph 2 	*Speed correct, directionality missing* – the sprinter shot off at tremendous speed *Directionality correct, speed incorrect or incomplete* – the train left the station and gained speed quickly *Speed incorrect or incomplete, directionality missing* – the cat was running faster and faster – the train started really slow, got faster then...		*Speed correct, directionality incorrect or incomplete* – the train came back from the station at a constant speed *Speed incorrect or incomplete, directionality missing* – a car increasing its speed – the train constantly sped up
graph 3 	*Directionality correct, speed incorrect or incomplete* – the train left the station. After 2 miles it started moving consistently.		*Speed incorrect or incomplete, directionality missing* – the train sped up and then came to a stop

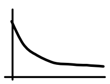
graph 4 	*Both descriptors correct* – the train left the station fast then started to slow down – the train left the station fast then started to slow down – the train started out fast then went at a regular pace *Directionality correct, speed incorrect or incomplete* – the train left from the station and moved at the same pace – the train slowly sped up while leaving the station – the train leaves a station but stops on the way out – the train left the station then went in a constant velocity – the train left the station then slowly sped up – the train left the station speed gradually – the train increased in speed as it left the station *Directionality correct, speed missing* – the train left the station *Speed incorrect or incomplete, directionality missing* – the train quickly gained speed – the train increased its speed and then	*Both descriptors correct* – the train goes slowly away from the station then slows down to a stop – the train went fast away from the station then started to go slow – the train moved slowly away from station and then slowly came to a stop – the train left the station fast then slowed down – the train left the station at a constant rate then slowed down as it got further away – the train started off moving fast away from the station then slowed down – the car left the house going pretty fast. when the tires started to deflate but the car did not stop, just went very slow – the train left the train at a steady pace and then slowed down – the train left the station fast then gradually started to slow down – the train left the station fast then started to slow down – the train left the station pretty fast and then gradually slowed down – the train moved fast from the station then slowed down – the train left the station fast and then slowed down – the train left the station at a quick pace then slowed down – the train moved away from the station but started to slow down – the train left the station fast then slowed – the train left the station quickly then slowed down – the train moved away from the station fast, but then slowed down – the train left the station and slowed down	*Both descriptors correct* – the train left the station fast then slowed down – the train leaves but starts to slow its pace – *Directionality correct, speed incorrect or incomplete* – the train slowly left the station – train left station at a inconsistent rate *Directionality correct, speed missing* – the train left *Speed incorrect or incomplete, directionality missing* – the train started off slow then increase the distance – the train sped up at an increasing rate – the train went very slowly *Both descriptors incorrect* – the train is quickening up then slowly evens out – the train started and accelerated and stopped *Reference to something other than speed or direction*

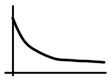
	continued at a constant speed *Directionality incorrect, speed missing* – the train came back to the station *Both descriptors incorrect* – the train is coming to the station at a steady rate – coming fast toward station – the train was coming in the station at a constant rate *reference to something other than speed or direction* – the train was moving in a curved motion – the sprinter started to level out and stopped – the train curved right and sped up – the train plunges off the track – the train started to leave and took a wrong turn	– the train started out going fast then slowed down *Speed correct, directionality incorrect* – the train started fast then slowed down to easily get to the station *Speed correct, directionality missing* – left fast then slowed down – fast then slower and slower – the train started fast then slowed down to easily get to the station – the train sped off the tracks but then slowed to a stop *Directionality correct, speed incorrect or incomplete* – the train is leaving faster and faster – the horse gunned it out of the cages and then got faster and faster – the train leaves the station very fast – the train moved farther away and sped up while doing this – I started off at an increasing running pace and then became tired and slowed down *Speed incorrect or incomplete, directionality missing* – the train started out slow and got faster on	– the plane moved down the runway then took off

		the way – the train is gaining more speed over time *Reference to something other than speed or direction* – the speed of the car – the biker increased speed up the hill – (the train sped off the tracks but then slowed to a stop)	
graph 5 	*Both descriptors correct* – the train leaves the station slowly and then speeds up – the train left the station, then drove faster and faster – the train left the station and picked up speed – the train kept speeding up as it left the station – the cat started off slow and then sped up – the train started out slow then went fast *Speed correct, directionality incorrect* – the train gradually sped up while curving left *Speed correct, directionality missing* – the train went from a stop to a faster speed – the speed slowly increases – the train sped up quickly – the train was stopped and then increased speed	*Both descriptors correct* – the train left the station and its speed increased every mile – the train left the station getting faster and faster *Speed correct, directionality missing* – as time increased the speed of the car increased	*Directionality correct, speed incorrect or incomplete* – the train left the station and changed speed *Both descriptors incorrect* – the train was coming into the tracks at a slow pace *Reference to something other than speed or direction* – the train was at the station and broke down

	*Directionality correct, speed incorrect or incomplete* – the train left the station and gradually slowed down – the train left the station at a very fast pace – slow from station *Both descriptors incorrect* – the train was coming into the station and slowed down on the way – the train stopped then went at a ?? rate – the train traveled little distance in a long time *Reference to something other than speed or direction* – (the train gradually sped up while curving left) – the train started off the track and got slower – the train started going backwards – the train is turning slowly back to the station		
graph 6 	*Directionality correct, speed incorrect or incomplete* – the train is changing speed to the station – train came into the station at a declining rate – slows down to the station	*Both descriptors correct* – train was coming in the station fastest *Reference to something other than speed or direction*	*Speed correct, directionality incorrect* – train left part at a constant rate – the train left the station at a constant speed *Speed incorrect, directionality missing* – slowing down really fast

	*Both descriptors incorrect* – the train was moving faster and faster away from the station *Reference to something other than speed or direction* – the train arrived on time but it crashed – the train moved very fast up hill	– the plane slowly descended downwards	– the train skidded to a stop *Reference to something other than speed or direction* – the train went up to down and kept going – the train was going down a hill – the train went into reverse
graph 7 	*Reference to something other than speed or direction* – the train was changing courses	*Speed correct, directionality incorrect* – the train left the station at a steady rate then stopped *Directionality correct, speed incorrect or incomplete* – came into the station and stopped in front of it	*Speed incorrect or incomplete, directionality missing* – the train was slowing down then stopped *Speed incorrect or incomplete, directionality incorrect* – the train started at a constant rate, stopped and then went on – the train started at a constant rate slowed to a stop and continued
graph 8 	*Both descriptors correct* – the train slowly slowed down coming into the station. – the train was coming into the station fast then started to slow down – the train slowed, as it approached the station – the train was approaching the station and started to slow down. – the train was coming into the station quickly	*Both descriptors correct* – the train headed toward the station and got slower and slower to a stop – the train got slower and slower as it came into the station – the train was going to the station fast and then slowed down – while going into the station the train got slower and slower	*Both descriptors correct* – the train came into the station gradually slowing down – lucky as the train approached the station it ran out of gas till it stopped *Speed correct, directionality missing*

	then started to slow down – the train keeps slowing down as it reached the station – the train slowed down and came to a stop at the station – the train was coming in at a fast speed then slowed to a stop *Speed correct, directionality incorrect* – started quick from the station then slowly slowed down. *Speed correct, directionality missing* – the speed slowly decreases – the train slowly decreases *Directionality correct, speed incorrect or incomplete* – the train was coming to the train to the same pace then dramatically sped up – the train quickly came into the station *Directionality correct, speed missing* – the train entered the station *Both descriptors incorrect* – it is leaving the train station at a steady rate – when the train left the station and the more	– the train sped toward the station then slowed as it approached – when the train was going back to the station it slowed down – the train came into the station fast then slowly slowed down – the train started off on a fast pace and slowed down as it go to the station – the train was going fast to the station then started to slow down but did not stop – the train came into the station fast started to slow down and then stopped – the train came into the station quickly then slowed down – the train moved fast toward the station then slowed down – the train was coming into the station quickly and then slowed down – the train came into the station at a fast rate but then slowed down so it would not crash – the train came into station fast then slowed down – the train came back going fast and then slowed down – the train was coming into the station and slowed almost to a stop gradually – the train came in and started to slow to a stop not at the station – the train came to the station fast, but it slowed down – the train approached the station fast then started to slow down *Speed correct, directionality incorrect* – the train left the station very fast. then slowed down *Speed correct, directionality missing*	– the train is slowly stopping then stops – slowly slowing down – the train kept going and went a little slower – the train steady slowed down – the train was going fast then slowed *Directionality correct, speed incorrect or incomplete* – the train came into the station at an increasing rate – the train slowly came to the station – the train started far away and gradually came in *Speed incorrect or incomplete, directionality missing* – the pace quickly picks up then starts to slow – a car slowly increasing its speed in a decreasing pace *Both descriptors incorrect* – the train left the station and changed speed *Reference to something other than speed or direction* – the train dipped low ? to wrong turn then kept going

	the train moved away the faster it went – slowed down and keep steady pace *Reference to something other than speed or direction* – the train was moving in a curved motion – ...stopped because it hit a rat – the train traveled far in a short amount of time – the train was coming in and then decided to go up onto a different track.	– the train is slowing down on the tracks – the runner slowed down at a quick rate – the car was decreasing then stopped – fast then slower and slower *Directionality correct, speed incorrect or incomplete* – the train went slow towards the station then went fast – the train is moving faster as it gets closer – the train started 5 miles away from the station and came toward the station faster and faster – the train got closer to the station and remained same speed *Both descriptors incorrect* – the car was driving at 75 mph and then it suddenly slowed down to 15 mph then continued to go at the same speed *Reference to something other than speed or direction* – I slid down the top of the hill and flew down until it became flat on the bottom of the hill	

Study 1, Part 2: Written Response Scores

Algebra I	Pre-	Post-
Student 1	1	6
Student 2	2	6
Student 3	2	4
Student 4	0	5
Student 5	2	3
Student 6	0	0
Student 7	0	5
Student 8	2	6
Student 9	0	2
Student 10	0	6
Student 11	6	6
Student 12	1	6
Student 13	0	6
Student 14	NR	2
Student 15	4	6
Student 16	6	6
Student 17	0	6
Student 18	4	1
Student 19	0	3
Student 20	4	6

Algebra I	Pre-	Post-
Student 21	3	6
Student 22	0	4
Student 23	0	NR
Student 24	2	4
Student 25	6	6
Student 26	0	5
Student 27	0	2
Student 28	4	2
Student 29	3	6
Student 30	2	6
Student 31	2	2
Student 32	4	5
Student 33	2	1
Student 34	3	1
Student 35	6	6
Student 36	5	2
Student 37	4	5
Student 38	4	4
Student 39	1	6
Student 40	0	6

Algebra II	
Student AII 1	2
Student AII 2	4
Student AII 3	3
Student AII 4	4
Student AII 5	3
Student AII 6	0
Student AII 7	2
Student AII 8	2
Student AII 9	0
Student AII 10	3
Student AII 11	4
Student AII 12	3
Student AII 13	2
Student AII 14	2
Student AII 15	2
Student AII 16	0
Student AII 17	0
Student AII 18	0
Student AII 19	1
Student AII 20	0
Student AII 21	NR
Student AII 22	NR
Student AII 23	0
Student AII 24	0
Student AII 25	NR

NR = No response given on testing

Appendix D

Study 2, Part 1: Individual Scores for Intervention and Control Group

Intervention Group

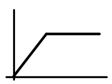
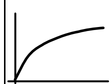
Student 001	5
Student 002	8
Student 003	8
Student 005	12
Student 007	6
Student 008	12
Student 009	12
Student 010	12
Student 014	10
Student 015	6
Student 016	12
Student 017	12
Student 018	12
Student 020	6
Student 023	12
Student 027	8
Student 028	10
Student 040	12
Student 041	10
Student 047	12
Student 049	12
Student 050	12
Student 051	12
Student 052	12
Student 054	8
Student 055	12
Student 081	12
Student 082	12
Student 083	10
Student 084	12
Student 085	12

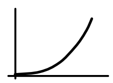
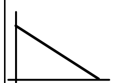
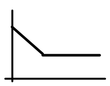
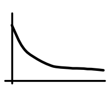
Control Group

Student 101	10
Student 102	10
Student 103	8
Student 104	7
Student 105	7
Student 106	8
Student 107	12
Student 108	12
Student 109	12
Student 110	2
Student 111	2
Student 112	12
Student 113	12
Student 114	2
Student 115	12
Student 116	4
Student 117	5
Student 118	6
Student 119	10
Student 120	9
Student 121	2
Student 122	5
Student 123	3
Student 124	12
Student 125	10

Study 2, Part 2: Written Responses for Intervention and Control Group

Table of written responses by student

	Algebra 2 Intervention	Algebra 2 Control
graph 3 	– the train is moving	– the train left at a constant rate then stopped – train left station, and traveled fast – train leaves gain speed and then goes consistent
graph 4 	– the train was heading towards the station while slowly decreasing speed – the train left the station at a fast rate and got slower – the train started to slow down as it got closer to the station – the train left the station fast and gingerly slowed down – the train was moving away from the station slower and slower – the train slowed down as it approached the station – the train was getting faster as it moved away from the station then slowed down – the train left the station and then crest in speed but leveld out after a while – sarah started riding her bike up a tough hill and started to slow down. – a train leaves a station slowly at first and gradually picks up speed – It left the station at a steady/speedy rate – the train left the station at a fast rate of speed then began to slow down – the train moved away from the station slowing down – the train left the station quickly and gradually slowed down – the train left the station and slowed over time to a stop – the train left the station quickly then slowed down – the train started at the station at a fast pace then slowed down – the train traveled fast away from the station but then over time slowed down – the train was coming into the station but started to lose gas – the train moved away from the station at a fast pace and then moved at a steady rate	– the train kept increasing its speed – the train started slow then started to get faster and faster – the train left slowly then gained speed – the train was gaining speed a lot at the beginning ? was rising little by little – the train started at the station then went up at a constant speed – the train left the station and got slower as it went – the train left the station and slowly got slower – the train left the station and increased speed – the train left the station at a slow rate then moved faster and faster – the train slowly left the station – the train slowly left the station – train left station at a slow rate – a train was moving at a steady rate sometimes turning – at first the train booked it and then slowed down – the train pulled away from the station and ? at the second stop – the m3 accelerated away from the station at an increasing rate – the train left the station at a constant pace then stopped – train leaves and gains speed – a train left the station constantly getting faster – the train left the station at an increasing speed then slowed down

	– the train left the station quickly then started to slow down – the train left the station at a high speed and gradually got slower and slower – the train is leaving the station and slowly moving away – the train quickly accelerated then came to a constant speed – the train left the station quickly but then slowed	
graph 5 	– the train went faster and faster towards the station – the train was leaving slowly then went fast – the train started to move slow out of the station , then gained momentum	– the train left slow and sped up quickly – the train quickly left the station – the train left slowly then gained speed – the train was coming to the station slower and slower – the train left the station and increased speed – the train left the station at a constant pace then sped up
graph 6 	– the train came to the station at a constant rate – the train slowed down as it moved closer towards the station – the train is coming into the station at a constant speed	– the train left the station at a constant rate – the train crashed after leaving the station – the train was coming back to the station
graph 7 	– the train left at a constant speed then it slowed down – the train was going toward the station and stopped	– the train came in very quickly and suddenly stopped – the train left the station and eventually settled on a constant rate
graph 8 	– the train is moving – the train left the station at a fast pace and then slowed down – the train moved away from the station at a constant fast pace – the train came into the station fast but then slowed down – the train was heading towards the station fast then it slowed – the train was coming in the station fast and then slowed down – the train was coming into the station fast then slowed down – the train sped up as it approached the station – the train was coming into the station and slowing down – the train was coming to the station and decreasing in speed as it gets closer – Sarah cam back from her bike ride and went down the hill faster and faster and started to apply her brakes. – a train slowly approaches a station and stops before it gets there – the train was coming into the station fast but began to slow down – the train was slowing down as it came to the station – the train came back to the station and slowly stopped	– the train slowly came into the station – the train left the station and kept lowering its speed – the train started fast then decreased the speed to the station – the train left the station and went slower and slower – the train was slowing down then to a slow and steady pace – the train is coming into the station coming slower and slower – the train was coming back to the station getting slower – the train is coming into the station slower and slower – the train slowly arrive at the station – the train slowed down as it came into the station – a train was moving at a steady rate, turng at the same time – the train started to go slow – the m3 decelerated from the station at a constant rate – a train came into the station constantly getting slower – the train left the station and slowed down at an increasing rate

	– the train gradually came towards the station and slowing down – the train came into the station fast but then slowed down – the train slowed down gradually into the station – when the train was coming to its stop it slowed down – the train began to head back to the station and began slowing down at a steady rate – the train was coming into the station at a fast speed then slowed down – the train was coming into the station at a high speed and gradually slowed down – the train was on a constant decline the speed coming into the station 'til traffic made it stop – the train was coming in the station fast then slowed	

Appendix E

Study 3, Part 1: Individual Scores for Intervention and Control Groups

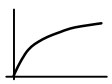
Intervention Group

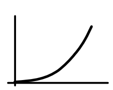
Student 301	12
Student 302	12
Student 304	12
Student 305	12
Student 306	12
Student 308	9
Student 311	12
Student 312	11
Student 314	10
Student 315	12
Student 316	12
Student 317	9
Student 319	7
Student 320	7
Student 321	12
Student 322	12
Student 323	9
Student 324	12
Student 325	12
Student 326	9
Student 327	9
Student 328	6
Student 329	12
Student 331	12
Student 332	12
Student 333	6
Student 334	12
Student 335	12
Student 336	12
Student 337	12
Student 338	12
Student 339	12
Student 340	12
Student 341	10
Student 342	12
Student 343	12

Control Group

Student 352	10
Student 353	12
Student 354	11
Student 355	12
Student 356	7
Student 357	12
Student 358	10
Student 359	7
Student 360	12
Student 361	6
Student 362	12
Student 363	12
Student 364	9
Student 366	9
Student 367	10
Student 368	9
Student 369	12
Student 371	12

Study 3, Part 2: Written Responses for Intervention and Control Groups

	Intervention Group	Control Group
Question 7 	Level 3 responses – the number of people with the flu increased very fast at first but then slowed down as the days went on – as the days went on less and less people got the flu – At first people rapidly got the flu and then less and less people got it each day – the number of flu out breaks lessens as the day goes on – with each passing day the number of people with the flu increases slower and slower – less people caught the flu every day Level 2 responses – Sarah had the flu and she went to a party so over the next couple of days all her friends got sick – The number of people with the flu increased every day – as the days went on more and more people got the flu faster and couldn't get rid of it – as days passed more people got the flu – the number of people with the flu are increasing slowly over the days – as winter came more people and more people got the flu – the number of people with the flu increases as the days increases – as days go by people will increase the flu – over time more people spread the flu around – the number of people who got the flu increased – more cases of flu across the country every day – as the days increase more and more people got the flu – as the day increased the number of people with the flu increase faster and faster – as each day went by the number of people with the flu increased – when Marcy caught the flu a day later the domino affect began, each day a couple of people caught it. Level 1	– the more days the flu is out the number of people increase at first then slow down – less people catch the flu every day – the number of people with the flu increased less and less each day – as the days went by the number of people with the flu increased slower and slower – everyday the amount of people with the flu went up slower Level 2 responses – a small amount of people at school got the flu, then other people catch it, quickly more people got sick – as the days went on the number of people with the flu increased quicker and quicker – the number of people that have the fu increase day by day – people with flu increase at a slow rate each day – more people got the flu each day but then leveled out to a constant rate – Level 1 – the more people sick the more days also – as the days got colder the increase of people with the flu increase more and more then stayed at a consistent rate – as days went on slowly continued getting the flu Level 0 – as the days increase, the number of people with the flu decreased. – the more people that got the flu, the more days they didn't go to school. – as the days increase the number of people with the flu decrease – the number of people with the flu has decreased by the days – the hall got emptier as it got later

	– as the number of flu increased the number of days did – As the number of people with the flu increases, the days without students increase – as more and more people get sick the more days they miss – as more people got the flu, the more days people were out Level 0 – As the number of days increased, the number of people with the flu decreased – as the days increased less people have the flu – As days went by the number of people with the flu grew smaller – the number of people who got sick the longer it took for people to recover – as the temperature dropped (days) the amount of people with the flu rose – the water went up quick then slowed to a steady rate – the number of people with the flu went down after some days – as people get better there is a constant rate – the number of days of the flu increases – as the days go on the flu is being attacked faster and faster – can't read it someting about ??more and more money	
Question 8 	Level 3 question 8 – As the ocean got warmer people stopped swimming fast at first – as the ocean rapidly heated the people came out of the water Level 2 – As the temperature of the ocean increased the number of people swimming decreased at a fast rate – As temp increased number of people swimming went down faster and faster – the number of people decrease slowly with the temperature of the ocean – the number of people swimming decrease as the temperature increased – the number of people swimming got higher but the temp went lower – as the ocean heats up less people choose to swim Level 1	– as the temp of the ocean rose the amount of people leaving slowed down Level 2 – as more temperature of ocean enter the number of people swimming decrease – as the temp increase the number of people decreased Level 1 – the colder the water the less people swim – less people swim as the temperature of the ocean drops – colder water=less people; warmer water=more people – it was summer and many people went to the beach, a large amount of people went swimming. later on the water got colder and not as many people went swimming. – as the temperature of the ocean got colder the number of people swimming decreased more and more – when the ocean water is cold the number of swimmers decreases

	– As the temperature of the water decreases, the number of people swimming decreases then stays the same – As the temp of the ocean decreased the people in the ocean left – the number of people swimming decreased as the ocean temperature decreased – as the ocean got colder people rapidly stopped swimming – as the temperature went down the more people got out of the ocean – As the temperature of the ocean decreases so does the number of people swimming – as the temperature of the water decreased the number of people decreased – when the ocean got warmer the more people went in – As the temperature dropped in the ocean, less people were swimming – he number of people decreases temperature will decrease – as the temperature of the ocean decreases, the more people leave the water – warmer the water the more people swim – as the temperature of the water decreases the number of swimmers decrease slower and slower – as the temperatures of the ocean decreased, the number of people decreased too. – as the number of people in the pool decreases the temperature of the water increases – as the temperature of the ocean drops so does the amount of people – as the temp of the water decreased so did the people swimming faster and faster – less people swam as the temperature got cooler – the less people swimming in the ocean the temperature of the ocean goes down. – As the temperature in the ocean decrease slowly people stopped swimming Level 0 – everyone was swimming loved it until the sun went away and made the water cold – As the number of people swimming decreases, the ocean temp	faster and faster – the people swimming in the ocean decreases each day as the the temperature goes down Level 0 – as the number of people slowly decreased, the temperature of the ocean became constant slowly – the colder the ocean the less people went swimming – the people swimming in the ocean decrease at a consistent rate because of the temperature of the ocean – the number of swimmers decrease slower and slower as the temperature changed – as the water got better the number of people fastly got out of the ocean – the number of people swimming the temperature of the ocean decreases – as the temp of the ocean changed the number of people swimming decreased – the party ?? bumping at first then everyone left

	evens out – as people swam the lower the temperature was – as clouds came over the ocean some people stopped swimming – the water dropped fast then slowed – as people swimming get cold they get out – the temperature of the ocean got colder – cant' read it something about losing a lot of money then it ?? flat	